Journal of Pedagogical Research
Volume 6, Issue 3, 2022
https://doi.org/10.33902/JPR.202213737

Research Article

Zero in arithmetic operations: A comparison of students with and without learning disabilities

Dilşad Güven Akdeniz[1], Esra Selcen Yakıcı Topbaş[2] and Ziya Argün[3]

[1]Mathematics Education, Bayburt University, Bayburt, Turkey (ORCID: 0000-0001-7387-5770)
[2]Mathematics Education, Gazi University, Ankara, Turkey (ORCID: 0000-0003-0554-6312)
[3]Mathematics Education, Gazi University, Ankara, Turkey (ORCID: 0000-0001-8101-7215)

The aim in the current study is to examine the conceptualizations of zero in arithmetic operations among students with learning disabilities (LD) and no learning disabilities (N-LD). The similarities and differences in the understandings of students with LD and N-LD of zero in arithmetical operations will be discussed. The study is a multiple case study with a qualitative research design. Six students, 3 students with LD and 3 with N-LD aged between 10 and 12 years participated in the study. The data were collected through clinical interviews. The data were analyzed by content analysis. Students' limited understanding of zero and operations affects their interpretation of arithmetic operations with zero. The conceptualizations of students with LD and N-LD regarding specifically division by zero show similarities with the exception of their use of the knowledge about operations. It has been observed that LD students have developed a different algorithm when it comes to addition and multiplication with zero. Through this study examining the differences in the understandings of students with LD and N-LD on a specific concept in terms of underlying conceptions, it is thought to provide an insight in terms of discovering LD and a more detailed recognition of these students' mathematics.

Keywords: Learning disabilities; Arithmetic operations with zero; Clinical interview; Mathematics education

Article History: Submitted 4 November 2021; Revised 31 March 2022; Published online 3 June 2022

1. Introduction

Throughout history, zero has been interpreted philosophically and symbolically as different from other numbers. It is within the basis of mathematics although it represents nothingness (Gullberg, 1997). Understanding zero and showing the absence of a quantity or a multitude and how it is used are different from understanding other numbers (Reys, 1974). Moreover, zero is quite different from other numbers with its place in basic arithmetic operations along with its meanings representing a symbol and nothing. It is special in terms of its place in presence and absence. Assigning a name to a non-existing object is a constraint in the historical process (Seife, 2000). It is also stated in the literature that students' knowledge of the concept of zero is weak (Grouws & Reys, 1975; Reys, 1974; Reys & Grouws, 1975; Tsamir et al., 2000). It is well known that learning disabilities (LD) students generally have difficulties in grasping the symbols representing numbers

Address of Corresponding Author

Dilşad Güven Akdeniz, PhD, Bayburt University, Faculty of Education, Department of Mathematics Education, 69000, Bayburt, Turkey.

dilsadgvn@gmail.com

How to cite: Güven Akdeniz, D., Yakıcı Topbaş, E. S., & Argün, Z. (2022). Zero in arithmetic operations: A comparison of students with and without learning disabilities. *Journal of Pedagogical Research, 6*(3), 27-52. https://doi.org/10.33902/JPR.202213737

and performing activities that require manipulation of these symbols such as addition and subtraction (Mejias et al., 2012). Examining understanding of zero in arithmetic operations among LD students who indicated that they have difficulty in four operations (Geary 1990; Gersten et al., 2005; Gersten et al., 2008; Namkung & Peng, 2018; Siegler & Shrager, 1984) would be useful in revealing the students' errors and misconceptions about operations. In fact, for the effectiveness of intervention studies conducted with LD students, first of all, the differences in their understanding should be revealed. It is thought that the concept of zero can provide a strong ground for this and provide different perspectives on the teaching of operations. Considering the concept of zero, which is generally investigated in the context of the division operation in the literature (e.g., Crespo & Nicol, 2006; Grouws & Reys, 1975; Reys & Grouws, 1975; Tsamir & Sheffer, 2000), in the framework of all four arithmetic operations can provide insights regarding the concept and operations. In this context, the aim of the present study is to examine the understandings of students with LD and no learning disabilities (N-LD) of the concept of zero in arithmetical operations. In the study, the similarities and differences in the understandings of students with LD and N-LD of the concept of zero in arithmetical operations will be discussed.

1.1. Theoretical Framework

1.1.1. The Concept of Zero

It took a long time for societies to confront, represent, and include the idea of zero (Sarama & Clements, 2009). The reason for this difficulty may go beyond assigning numbers to a series of objects; beside the point that zero differs from other numbers with its meaning, it is a number that causes problems in basic arithmetic operations that students encounter in primary school. Zero, which is the identity element of the addition operation in number systems and the only element of the rational numbers that does not have an inverse element in multiplication, is an exception by behaving differently from other numbers in the basic arithmetic operations that these students face. The fact that a student with the idea that addition and multiplication operations increase the number becomes aware that adding zero to a number does not change the number and that multiplication of a number by zero results in zero may provoke a cognitive conflict that must be turned into an opportunity. Zero is not a counting number unlike the other natural numbers, yet it has the feature of being a placeholder. Thus, understanding the concept of zero means understanding the meanings and uses of the zero number and the reasons behind the algorithms and operations for this and understanding the concepts of number, place value, and placeholder. The related literature includes some studies supporting the findings reported by Inhelder and Piaget (1969) showing that students cannot grasp zero as expected until formal operations are acquired (e.g., Blake & Verhille, 1985; Catterall, 2006; Reys & Grouws, 1975; Seife, 2000; Tsamir et al., 2000; Wheeler & Feghali, 1983). Understanding the number zero and its use as a counterpart supports understanding of all these concepts and algorithms that are essential for primary education. However, studies have shown that students' knowledge of the concept of zero is weak (Grouws & Reys, 1975; Quinn et al., 2008; Reys, 1974; Reys & Grouws, 1975; Tsamir et al., 2000).

The related literature indicates that the concept of zero is investigated in terms of its interpretation in the division operation (Crespo & Nicol, 2006; Grouws & Reys, 1975; Reys & Grouws, 1975; Tsamir & Sheffer, 2000), whether it is an odd or even number (Levenson et al., 2007), with negative numbers in the transition from arithmetic to algebra (Gallardo & Hernández, 2005) or in the transition from natural numbers to integers (Gallardo & Hernández, 2006), and the context of an empty set (Merritt & Brannon, 2013). It can also be seen that the majority of the few studies examining zero with all its meanings and uses were carried out with student teachers (Wellman & Miller, 1986; Wheeler & Feghali, 1983). Therefore, there is a need for studies examining students' understanding of zero by including its meanings and uses in arithmetic operations. In the present study the understanding of students with LD and N-LD of zero in arithmetic operations was investigated.

1.1.2. Zero in Operations

Zero has a history shorter than that of other single-digit numbers. Since zero is a number, mathematicians reasoned they would be able to perform operations with zero. However, zero has caused different perspectives and some problems in arithmetic operations (Joseph, 2008; Nath, 2012; Sen & Agarwal, 2015). Below the role and structure of zero in operations are described.

Zero in addition and subtraction. The concept of zero has an identity element role in addition. The reason for this comes from the meaning of adding, combining, and increasing multiplicities. When you add a zero to any number, combine the quantity of absence with the amount of the asset, or increase the amount of the asset by the amount of absence, you will still get the amount of the asset. Therefore, where a is any real number, $a+0=0+a=a$.

Since every element in the set of natural numbers is not inverse with respect to addition, subtraction is defined as decreasing. In the subtraction process, which is defined as the opposite of addition, in the context of decrease, when you subtract itself from a number or when you subtract itself from an asset amount, the number you will get will be zero, which represents absence. The result of the operation will not change when the amount to be subtracted is zero in the set where the subtraction is defined, in other words, when you subtract a number that expresses absence from a quantity, that is, zero. Subtracting a number from zero makes sense when our number set is integers. This is understandable for students only at the secondary school level, where integers are introduced (Ministry of National Education of Turkey [MoNE], 2018).

Zero in multiplication and division. The role of the concept of zero in multiplication and division is critical. In accordance with the repeated addition meaning of multiplication, when we multiply zero by an a number or an a number by zero, the zero is added as the number a and the result is zero. Thus, a number multiplied by zero or zero multiplied by a number is zero. In other words, $ax0 = 0xa = 0$ for the number a. Therefore, in the multiplication operation, the absorbing element is attributed to the zero.

Things get more complicated when it comes to zero and division. Zero divided by zero, a nonzero number divided by zero, and zero divided by a nonzero number are important mathematical points that need to be examined.

- Mathematicians have made different claims in the past about the division of zero by zero. For example, famous Indian mathematician Brahmagupta claimed that $\frac{0}{0} = 0$ (Boyer & Merzbach, 2011) even though he knew that any number multiplied by zero is zero. In the equation $0xa = ax0 = 0$, the idea arises that the missing factor is uncertain because a is any number. Here, as a is a finite number, it can be said that the equality of $\frac{0}{0} = a$ is provided for all a values. However, it is not clear which value of a we will get. Hence, a can be any number. Therefore, $\frac{0}{0}$ is also indefinite since a is indefinite.
- The division of a non-zero number by zero has a similarly complex structure. Considering that division is the inverse of multiplication, the idea arises that no matter what number we multiply by zero, an a number other than zero cannot be obtained and such a number does not exist, that is, it is undefined. For $\frac{a}{0} = x$, $a = 0.x$ is contradicted by the fact that a is a non-zero number, so the operation $\frac{a}{0}$ is not meaningful and an undefined concept emerges.
- If zero is divided by a number, it is concluded that $\frac{0}{a}$ is zero, in accordance with the idea of equal partition, with a being a non-zero number.

1.1.3. Learning disability

An individual with a learning disability is described by the Ministry of National Education of Turkey as follows (MoNE, 2006):

> the individual in need of special education and support that arise in one or more of the information retrieval processes required to understand and use the written or oral language, and because of

> difficulties in listening, speaking, reading, writing, spelling, concentrating, or mathematical operations.

Upon examination, it can be seen that the definition used in the Individuals with Disabilities Educational Act is similar:

> "specific learning disability" means a disorder in one or more of the basic psychological processes involved in understanding or in using language, spoken or written, a disorder of which may manifest itself in an imperfect ability to listen, think, speak, read, write, spell, or to do mathematical calculations (Council for Exceptional Children, n.d).

Two of the 6 symptoms of learning disability identified by the American Psychiatric Association [APA] relate to the field of mathematics (APA, 2013). These symptoms are difficulties in perceiving numbers, in developing number facts or calculations, and about mathematical reasoning. Similarly, it is stated that students with LD in mathematics exhibit weaknesses in number-related processes and procedures (e.g., subitizing, symbolic number comparison, number line estimation) (Andersson & Östergren, 2012) and a 10-year-old student with a mathematics learning disability is at the level of 5-year-old child with normal development (Piazza et al., 2010).

1.2. Current Study, Rationale and Research Questions

Geary (1990) found that students with LD frequently make mistakes in calculating and retrieving from memory (difficulty in remembering very simple number combinations such as 7 – 5 or 3x7), use immature calculation strategies, and choose weak strategies. Students with LD may continue to count with their fingers when their peers have already abandoned this habit. Their strategies for addition operations may not be developed (for example, for 3+8, identifying 3 concrete objects first and then adding 8 concrete objects, and counting how many they are). For example, they may not use a stronger strategy such as starting the addition process by adding on 8 (Siegler & Shrager, 1984) or they rely on some inefficient counting strategies such as counting all numbers (1, 2, 3, 4, 5, 6, 7 for 3 + 4 = 7) (Namkung & Peng, 2018). They are not aware of the commutative property of addition and that the same result is obtained when they add 8 to 3 (Gersten et al., 2005; Gersten et al., 2008). Goldman, Pellegrino, and Mertz (1988) claim that these students can automatically perform these operations like their peers by using the technique of counting on (by taking 2 + 9 as 9 + 2). However, the performance of the students in routine applications, including the basic arithmetic operations, is not an indicator of mathematical thinking (Schoenfeld, 1994). Therefore, the failure of a student with LD in routine operations does not necessarily mean that s/he cannot think or will not be able to think mathematically. It is stated in the literature that LD students may have different understandings of mathematical concepts (Güven Akdeniz & Argün, 2018; 2021; Lewis, 2014). It seems important to examine these different understandings and thinking in depth in order for students to be better comprehended in terms of their mathematical learning. An in-depth analysis of the understanding of operations with zero by a qualitative study may provide an insight into LD students' understanding of the four operations. Although zero is a difficult concept to learn, it provides an opportunity for a profound reasoning process (Wellman & Miller, 1986). Therefore, through the present study examining the differences in the perceptions and thinking of students with LD and N-LD on a specific concept in terms of their underlying understandings, it is thought to provide an insight in terms of understanding LD and a more detailed recognition of these students' mathematics. In this sense, our study will aim to answer the following questions:

- What and how are the understandings of zero in arithmetic operations of the students with LD and N-LD?
- What are the similarities and differences in the understandings of the students with LD and N-LD of the concept of zero in arithmetical operations?

2. Method

2.1. Research Design

The present study, which thoroughly examines and reveals the similarities and differences in understanding of the concept of zero among students with LD and N-LD, is a multiple-case study with a qualitative research design (Yin, 2003). The existing insights of the students with LD and N-LD were revealed without any intervention (Stake, 1995; Yin, 2013). The case of the study consists of students with LD and N-LD. Students' understanding of the concept of zero is the unit of analysis in the study.

2.2. Study Group

The participants of the study were 3 students with LD (Mert, Serkan, Ilayda, the names are pseudonyms) and three students with N-LD (Elya, Kaan, Aybars, the names are pseudonyms). While the students with LD were in the 10 and 12 age range, those with N-LD were 4th grade students 10 years old.

The LD participants in the study were selected from among students who were determined to have LD by the local Counseling and Research Center. Those in the N-LD group, on the other hand, were chosen from among students with high achievement level, taking the opinion of their teachers in account, to ensure no chance of LD. All of the students were living in the same city but attending different schools. The LD students were attending the same special education center. The LD students were contacted through the special education center they were attending.

Participants were selected by criterion sampling, a purposeful sampling method. As the research includes division by zero and division of zero in the category of various uses of zero, it was thought appropriate to include 4th graders in the study (see MoNE, 2018) due to the fact that zero-zero or zero-division operations were included in the study. However, considering the fact that the students with LD are two years behind their peers (APA, 2013; Judge & Watson, 2011), the LD students in the age range of 10-12 were chosen in order to use the same interview form as for the N-LD students. We took care to choose N-LD students at the age of 10 (4th grade), with similar achievement levels, not low (to ensure they do not have LD), who were not diagnosed with LD and not likely to have them. Due to the heterogeneous nature of the students with LD (Namkung & Peng, 2018; Pierangelo & Giuliani, 2006), we did not insist that they be at the same grade level since they may differ in terms of academic achievement despite being at the same grade level. For this reason, students were selected on a voluntary basis, with approval from their parents.

2.3. Process

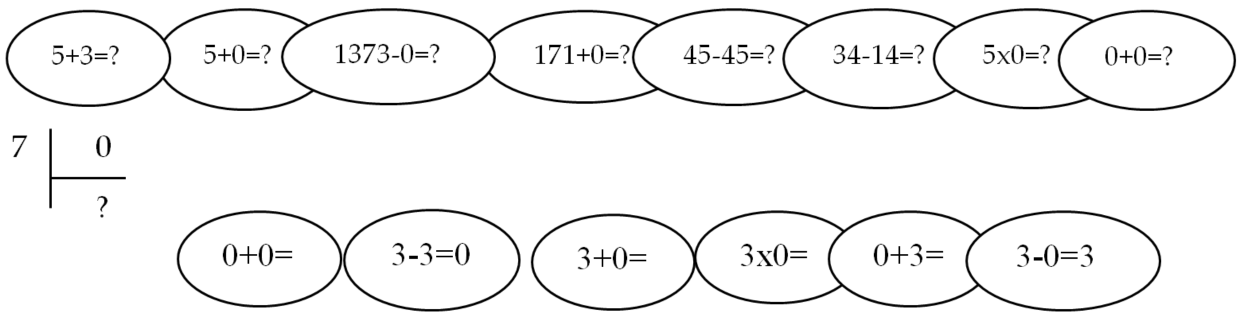
2.3.1. Data Collection

The data of the study were collected through clinical interviews aiming at revealing students' understanding of the concept of zero. Clinical interviews conducted at three separate sessions at most lasted in total between 60 and 120 minutes. The data were collected via camera and voice recordings, activity papers used in the clinical interviews, and scratch paper used by the students. The clinical interviews were carried out individually in environments that the students were accustomed to (Hunting, 1997). The interviews with the LD students were held in the classroom at the special education center the students were attending, while the interviews with the N-LD students were held at the students' schools. The interviews were conducted by the first two researchers. After each interview, the data were examined by the researchers and it was discussed whether an additional interview was necessary. Since it was deemed necessary, additional 10-15 minute interviews were held with two LD students.

Following the preparation of the clinical interview questions, pilot studies were conducted with two students who were not participants in the main study, one with LD and one with N-LD. For the clinical interview questions prepared as a result of the pilot studies, we obtained expert opinions from a primary school mathematics teacher and a mathematics professor.

Clinical interview questions. The clinical interview questions aiming to elucidate students' thoughts and understandings on the concept of zero in arithmetical operations were prepared according to the 4th grade level. A clinical interview form consisting of 11 questions was posed/adapted and applied to both student groups (see Appendix 1). The meaning of zero in operations was examined in terms of being the identity element in addition and the contradiction of this feature with the understanding that addition increases the sum, subtracting zero from a number, the feature of being the absorbing element in multiplication, and the contradiction of this feature with the understanding that multiplication always increases the result. Moreover, the state of uncertainty in division $(\frac{0}{0})$ (being the uncertain value), dividing a number other than zero by zero, the understanding of limit and unidentifiability (being the unidentifiable value), and dividing zero by a number were examined. These meanings, as suggested in the related literature (Duncan, 1971; Tsamir & Scheffer, 2000; Watson, 1991), were investigated in the context of problems that are associated with daily life to make students interpret the questions. For example, "There are no apples in a basket. What will be the result if I divide these apples into 4 groups?" or "I have 3 packages. There are no eggs in them. How many eggs do I have then?" As another example, it was anticipated that the students would not be able to directly reach the limit regarding the amount of zero within a number when the division of a number by zero was only asked in a symbolic context or directly. Therefore, the division of a number by zero is also addressed in the context of measuring a certain length with a 0-unit ruler. Thus, it was indirectly investigated whether the students have an understanding of limit. Some meanings and uses were also asked directly. For example, "What does it mean to divide zero by zero? By what number would I multiply zero to get zero? Can we divide zero by 9? What does this mean? Can we divide zero by 747? What does this mean? What will be the result?" Some uses and meanings of zero in operations were also questioned in a symbolic context. For example, the reasons for the results and the uses and meanings of zero were discussed by presenting them a series of questions (see Figure 1). For instance, for $3 - 3 = 0$, the questions "Why are the results like this? What does zero say here? What is the function of zero here?" were asked.

Figure 1
Sample questions directed to the students



Pilot studies, with an LD and a N-LD student were conducted under the same conditions as the participants and expert opinions were obtained before the interviews. As a result of the pilot studies and expert opinions, some questions were changed, some were removed, and new prompt questions were added. For example, in a symbolic context the question "0-123=?" had been asked in the pilot study but it is not meaningful for 4th grade level students especially N-LD students, so it was removed. The interview questions are presented in Appendix 1.

Different tasks and questions were prepared by considering the four operations with the concept of zero and the underlying conceptual facts (such as the measurement meaning of division). For example, the process of adding with zero was first questioned in the context of "add, increase" verbally, and then it was questioned in a symbolic context. The emphasis on the concept of zero in the primary school mathematics curriculum begins with the examination of the effect of zero on the addition process in the first grade. Then the learning outcome "It is shown that the difference of two equal natural numbers is zero" is included. Although there is no separate title for the concept of zero before, there is the acquisition of "Identifies the number of objects in a mass with up to 20 objects (including 20) and writes this number in numerals". It is aimed to determine multiplicity corresponding to a number up to 20 (MoNE, 2018). In this context, it is thought that teachers consider the number zero. However, during the introduction of numbers and operations with them in the curriculum, it is seen that there is no special remark for teachers or students about the number zero and operations with it.

2.4. Data Analysis

Content analysis was utilized for the data analysis. The clinical interviews were transcribed verbatim, and the transcriptions were read repeatedly. The data were analyzed by the first two authors separately between the groups. Then all the researchers came together, the data were read again, the codes and categories obtained by the researchers were discussed, and a consensus was reached when it resulted in different coding or categorization. After completing the data analysis, expert opinions were obtained from a mathematics educator regarding the reliability of the research data. The correlation coefficient between the researchers and the expert was calculated as 0.89 (Miles & Huberman, 1994).

First of all, the understanding of each student was examined separately for each of the four operations. The LD/N-LD students were then compared within themselves in terms of their understandings. In the last step, the LD/N-LD students' understandings were evaluated together to reveal differences and similarities. The documents of tasks and transcriptions were analyzed considering repeated patterns showing students' understanding of the concept of zero in four arithmetic operations. At this stage, mathematical knowledge about the place and role of the concept of zero in operations acted as a guide, and students' thinking and comprehension were coded in this context. Besides the known roles of zero in operations, the different meanings that students attribute to the number, to the property of the number, or the operations were also examined. Observations and patterns that were agreed upon as a result of the discussion were noted in the documents.

3. Findings and Interpretations

The findings are presented under four main titles: Students' understandings of zero in addition and subtraction, in multiplication, and in division and the similarities and differences in the understandings of the students with LD and N-LD. Further, the sub-categories obtained as a result of the data analysis are presented under the four main titles. Figures 2, 4, 6, 12, 13 and 14 were created based on the findings obtained in the research.

3.1. Students' Understandings of Zero in Addition and Subtraction[2]

Findings regarding the students' understanding of zero in addition and subtraction are as presented in Figure 2.

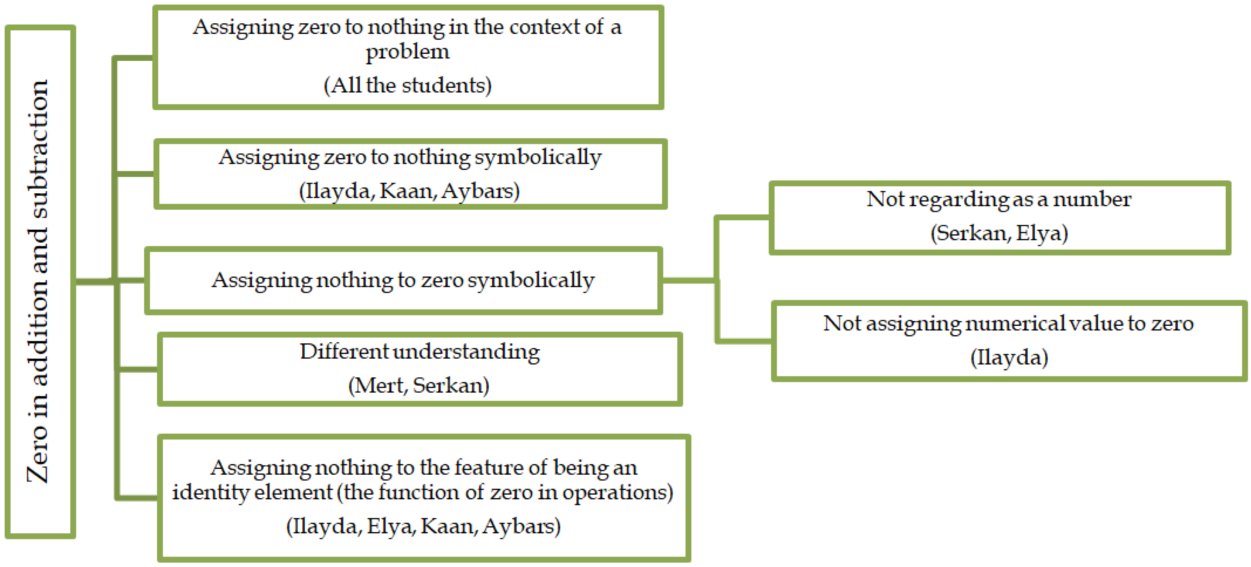
3.1.1. Assigning Zero to Nothing in the Context of a Problem

All participants had the right answer by assigning zero to nothing in daily life problems that require addition and subtraction with zero. For example, by giving the answer zero to the question "You have 3 biscuits, you will eat all of them, and how many biscuits will remain?", Serkan explained that "If you have 3 of them, it means you will have zero. This is because you ate them all. If there were 5 of them, you would have some remaining. But in this one, you ate all of them. If you left one, you would have one, but you ate them all. They are finished." Therefore, Serkan

[2]Addition and multiplication are mathematical operations while subtraction and division are respectively the opposite of these operations. However, in the present study, the terms subtraction and division are used as they are explained in the literature.

considered the situation of nothing reached by eating all the biscuits and stated that the result was zero.

Figure 2
Students' understanding of zero in addition and subtraction[3]



3.1.2. Assigning zero to nothing symbolically.

Ilayda, Kaan, and Aybars assigned zero to nothing in their symbolic addition and subtraction. When Aybars was asked to comment on the operation 3-3=0, he interpreted it as "I have three pieces of chocolate. I ate all three, so there are none, I mean zero". Similarly, the natural development process of zero started with assigning zero to nothing (Reys & Grouws, 1975; Seife, 2000). This is an expected perception of students who assign zero to nothing.

3.1.3. Assigning nothing to zero symbolically

Based on the idea that zero means nothing, Serkan, Ilayda, and Elya stated that zero was not a number and did not represent a value in the context of addition and subtraction, that is, they assigned nothing to zero.

Not regarding zero as a number. Elya with N-LD and Serkan with LD did not consider zero as a number in addition and subtraction. Since neither student conceptually accepted zero as a number, they did not consider zero as a number in the addition process presented in the symbolic context.

> Serkan (S): As we cannot add zero to zero, we write zero here.
> Researcher (R): We write zero as they cannot be added to each other?
> S: Yes.
> R: But we do addition here.
> S: If we add, there will be a number.

As can be seen, for Serkan, all negative conditions resulted in zero or if the operation resulted in zero; this meant that the operation could not be done and the result was zero since it could not be performed. Or, as zero meant nothing for Serkan, an operation could not be conducted only with zero, so the result was zero. Similarly, Elya expressed the idea that zero was not a number for the operations of addition and subtraction by saying "(for 0+0=0) we cannot have 1 by adding zero to zero, because if we add a number to a number, for example adding 1 to 1, we will have 2. But if you add zero to zero, the answer is zero *as zero is not a number*. For 3-3=0, if we subtract 3 from 3,

[3]Serkan and Mert were not included in the category "Assigning nothing to the feature of being an identity element" as a result of their different understandings of the role of zero in addition and subtraction. See the Different Understandings heading in this section.

the result is zero as none is left" regarding the meanings of zero in symbolic addition and subtraction with zero. It is noteworthy that Elya with N-LD and Serkan with LD have the same insights and the very same explanations for this.

Not assigning a numerical value to zero. Ilayda thought that zero did not have a numerical value in addition and subtraction. She interpreted its feature of being an ineffective element in this way. However, as seen in the following dialogue, she did not interpret its feature of being an identity element.

> Ilayda (I): In 3+0=3, zero is an identity element, I mean it is something like 3 erasing zero.
> R: Why does it erase zero?
> I: That is, zero cannot do anything to 3. This is because 3 is bigger than zero.
> R: When adding 1 to 3, the result is 4, but 1 is smaller than 3?
> I: But zero is an identity element, this is why.
> R: Why is it an identity element?
> I: This is because it does not have a value.

With these interpretations, it can be thought that Ilayda did not contradict the idea that zero was not a number. Although Ilayda was aware of the fact that zero was an identity element in addition, it is thought that her expressions such as "3 erases zero" are due to the fact that the function of zero in multiplication is called the absorbing element. As the students are introduced to the function of zero in multiplication as the "absorbing element", they may generalize the fact that the big fish eats the small fish to the operations of addition and subtraction.

3.1.4. Assigning nothing to the feature of being an identity element

Except for Mert and Serkan, the students were aware that zero did not change the result of the addition and subtraction (number minus zero). However, they differed in their answers to the questions of "Addition would increase the result, why did it not in this one? Subtraction would decrease the result, why did it not in this one?"

Ilayda, Elya, Kaan, and Aybars interpreted the fact that zero was an identity element with zero having no value and with nothing. For example, Elya explained the fact that zero was the identity element in addition and subtraction operations by assigning nothing to zero by expressing that "3+0=3, 3-0=3 because you take out nothing and don't add anything to this, zero is not a number, it is nothing, we have 3. I erase 3 things and I add zero here, but as zero is not a number to be added, the answer is 3."

Kaan directly assigned nothing to zero in the addition and subtraction operations with zero, and he attributed the feature of zero as the identity element to "nothing". For example, when he was asked "What if five and zero are summed? Does five increase?" he said "No, this is because zero is nothing. Subtracting zero from 1373 is like subtracting nothing from it".

Because Mert and Serkan evaluated the operations according to their own algorithms, the conversations about the ordinary algorithm could not be realized in a meaningful way. In their justifications for their own algorithms, they explained why the result was zero when zero was on the top or the result was the number on the top when zero was written at the bottom by considering its place in the operation (see the Different understandings heading). For instance,

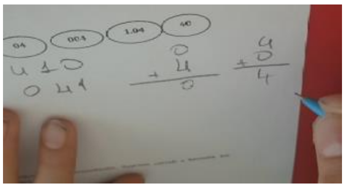
> Mert (M): 5+3, it is ok now, we add 3 to 5, the result is eight...
> A: Addition would increase the result, why is it not like that here (5+0)?
> M: This is because there is zero at the bottom there; therefore the number does not increase.

3.1.5. Different understandings

The algorithm examined under this heading is specific to the addition operations performed with zero. Mert and Serkan implemented the known algorithm in the operations conducted with other numbers; therefore, all the interpretations for the understanding of Mert and Serkan are specific to the operations with zero.

Figure 3

Mert's addition algorithm



The algorithm developed by Mert for the addition operation with zero was as follows: In Figure 3, the result is 0 because it is written on the top in the first operation. The result is 4 in the second operation because 4 is written on the top. In the 4+0 operation, the result is 4, and in 0+4, the result is 0. Mert's algorithm in the context of addition with zero was "the number on the top (with his words, it is "marked down") is the result":

M: 4 is on the top, adding to zero, the result is 4 [...] if zero was on the top, 4 would be marked down.
R: Why is it like that?
M: Because... this [zero or 4] is on the top, it is marked down.
R: Why is one of them [result] 4 and the other 0?
M: Well, I didn't think about that.

As it is seen, Mert was not aware of the commutative property of the addition process, so this may have caused him not to question his own algorithm. Further, the fact that the result of 4-0 is 4 may have caused him to consolidate and generalize this algorithm. For the subtraction and the operation of 3-0, Mert said that "it is directly marked down as zero is smaller". For the operation of 0-3, he said "This is on the top, so there will be no numbers", and when the reason was asked, he explained "This is because there are no numbers near this (zero)". Mert is aware of the fact that 3 cannot be subtracted from 0 in natural numbers. However, if there were another number next to zero, he could perform this operation, and, in the same vein, he found the result of the 10-3 operation to be 7. This may be due to the expressions used by the teachers in operations such as "in *a0-b* operation, *b* cannot be subtracted from zero, we will take a ten."

Serkan's algorithm ("the number on the top is marked down") was the same as Mert's algorithm in the addition operation. This overgeneralization by Serkan in addition operations may be due to his not regarding zero as a number (examined under the title of not regarding zero as a number). Unlike Mert, Serkan also applied this algorithm for subtraction.

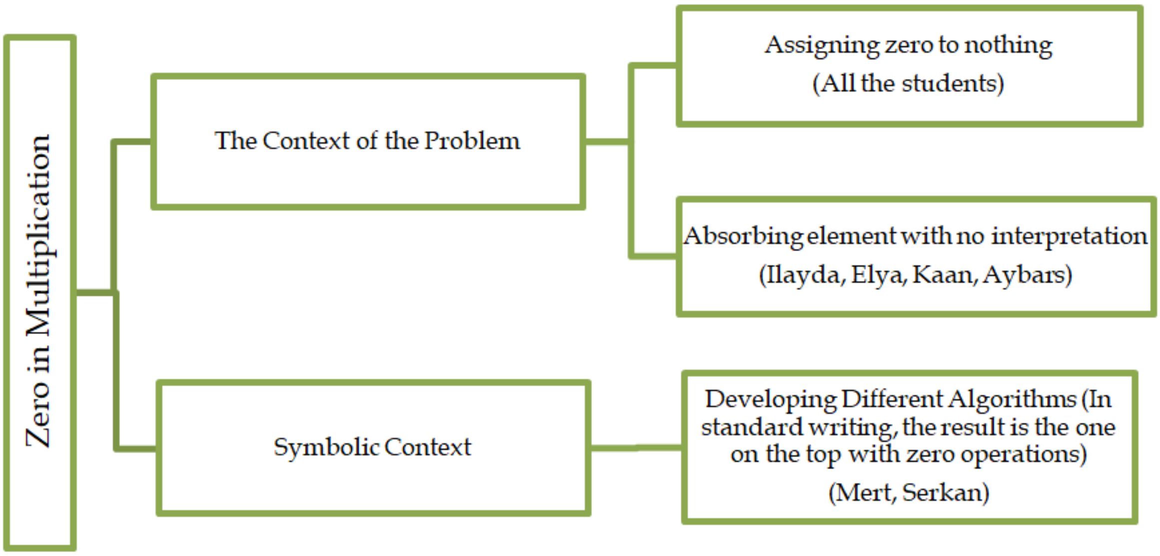
S: In 3-0, we subtract zero from 3. We do the same, as 3 is on the top, we mark it down.
R: What does it mean to you to subtract zero from 3?
S: ...
R: Let's say, you have 3 eggs, what does it mean to subtract zero from them?
S: If we have 3 eggs, they are gone. It means subtracting zero.

As can be seen, neither student made sense of the algorithms they conducted. They may relate the expression "*b* cannot be subtracted from zero in the *a0-b* operation (for example 20-3)" and the expression that a number cannot be subtracted from zero but zero can be subtracted from a number in natural numbers with zero written on the top or bottom in writing algorithms. Therefore, it can be claimed that Serkan overgeneralized this understanding to addition with zero. However, Mert did not make any comments about the result of the 0-2 process by indicating that the 0-2 operation could not be conducted in subtraction. Thus, it is difficult to comment on *why* Mert perceived the result as zero when zero was written on the top in the addition operation. However, this situation in subtraction with zero may reinforce this algorithm developed by Mert.

3.2. Students Understandings of Zero in Multiplication

Figure 4 illustrates the summarized findings regarding the students' understanding of zero in multiplication.

Figure 4
Students' understandings of zero in multiplication



3.2.1. Assigning zero to nothing in the context of the problem

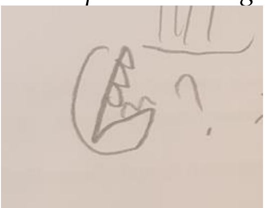
All of the students gave the answer zero by assigning zero to nothing in the multiplication operation asked in the context of the problem. For example, we asked the question "I have 3 packages. There are no eggs in these packages. How many eggs do I have then?" Kaan answered "You have none, that means you are cheated. You paid for none. You have 0 eggs." and Serkan said "Zero. Why? Because there are no eggs in the bowl, as it is empty, you have zero."

Absorbing element. As in the addition and subtraction operations, Serkan and Mert with LD applied their own algorithms for multiplication, and this algorithm did not have an absorbing element. Ilayda with LD and Elya, Aybars, and Kaan with N-LD were aware of the feature of the absorbing element but they did not make sense of this feature. For example, Ilayda was asked:

R: 5x0=0 can you give a similar example [in the context of a problem]?
I: Zero eats five… I cannot think of an example.
R: What does this result [in the operation 5x0=0] tell us?
I: The result will be zero when we multiply zero by a number
R: Why?
I: Because it is like zero is eating the numbers here.
R: What is the meaning of zero that we have here (the result)?
I: It is zero.
R: Ok this [showing the multiplier zero] ate it, but how do we get zero?
I: I don't know.
R: It eats and eats but doesn't it get bigger?
I: …

As this feature is called the absorbing element [swallowing element in Turkish] in the multiplication operations, it is thought that Ilayda expressed this as "swallowing or eating". Students may not attempt to explain multiplication by zero by considering the meaning of multiplication due to this naming. It is thought that the name given to this feature (absorbing element) does not reflect the conceptual meaning of the feature.

Figure 5
Zero representation given by Aybars with N-LD in the multiplication operation



Similarly, Aybars, Kaan, and Elya with N-LD could not make sense of the zero's feature of being the absorbing element. For instance,

E: It says 3x0, 0x3, 0.
R: Why?
E: Because it is the absorbing element
R: For instance, let's say that you don't know about this feature, then how would you find the result?
E: Can I think for a moment? [...] I think of addition. I don't know why. As I did not learn something like this, I don't know why. I cannot think of anything.

Although Elya associated the multiplication operation with the addition operation, she could not make sense of this feature of zero or the repetitive addition interpretation of multiplication; in short she could not come up with a conceptual explanation. As can be seen, the students did not have an in-depth understanding of the fact that zero was the absorbing element in the multiplication process.

3.2.2. Symbolic Context

Developing different algorithms (Algorithm created by Serkan and Mert). Serkan and Mert with LD developed an algorithm for multiplication by zero, similar to their algorithms in the addition and subtraction operations. These algorithms of students are valid for the operations conducted with zero. To briefly summarize the algorithm that Mert and Serkan developed for one under the other presentation in multiplication operations with zero, "the result is the number on the top" similar to the algorithm they developed in addition and subtraction operations[4]. When the reasons were examined, Mert stated that "the *a* on the top will be directly marked down" for the algorithm of *a*x*0*=*a*. Serkan stated that the result of the operation was the number itself because zero was not a number as in the addition and subtraction operations. It is noteworthy that the students get the same result and apply the same algorithms for different reasons. Considering that the educational backgrounds of the students (different schools, different teachers) are also different, this is an important finding. This situation may be due to the fact that the students misinterpret the algorithm and results by overgeneralizing an operation they encounter to the multiplication operation. They both applied their zero assignment schemes to the negative situations in the *0xa* process. However, Mert stated that he could not multiply because there was a bigger number on the top, while Serkan said that zero, which was not a number, could not be multiplied as it was on the top and thus the result would be zero (see Table 1).

In fact, although the reasons underlying the students' understandings are different, the reasons for the operations are not being able to perform the multiplication. As the operations cannot be performed as in an operation like *0xa*, they expressed that the result would be zero and the result is *a* in *ax0*.

[4] Mert did not always behave in a consistent manner in the multiplication of zero with two or more digit numbers, and made non-systematic errors.

Table 1
Multiplication operation algorithms developed by Mert and Serkan

	$ax0 = a$	$0xa = 0$
Mert	*Cannot be multiplied* M: 3x0=3 R: Why 3? M: Because they are not multiplied with each other, 3 is directly marked down. R: What do you mean by saying they are not multiplied? M: If there was one here, 3 would be marked down. If there was 2 here, multiply 3 with 2, it would be 6. But, as we have zero here, three is marked down.	*Assigning zero to the negative situation* (It cannot be performed if the bigger number is not on the top.) M: 0x3= 0 R: Why? M: Because it is not multiplied... I mean… we need to add a bigger number to it…
Serkan	*Zero is not a number* R: 3x0, S: [...] 3 is not multiplied by zero because if there was a number at the bottom, for instance we multiply 3 with 2, as there is 0 at the bottom, the result is 3. R: You mean 3 is not multiplied by zero? S: Yes. R: Why? S: As I told you, there must a number at the bottom. R: Why can we not do this like that? S: Because zero is not a number and the others are numbers.	*Assigning zero to the negative situation* R: 0x12 S: We cannot multiply, if 12 was on the top, we could. Therefore, we write zero.

It is noteworthy that the algorithm developed by students for the multiplication operation has similarities with the algorithm they developed in addition and subtraction operations. The reasons for Mert and Serkan to apply this algorithm can be their overgeneralization for the subtraction operation with zero. Therefore, Mert and Serkan did not use the feature of zero being the absorbing element in the multiplication operation. However, it is thought that the students used their overgeneralizations about the subtraction operation at that point. Serkan and Mert also confused subtraction with multiplication. They used the word "subtraction" while they were working on the multiplication operation. For example,

> S: 3x0, we cannot subtract zero from 3. Oh sorry, we cannot multiply 3 by zero because [...]
> M: [...] if there was one near it or another number, I would subtra… oh sorry, I would multiply. I sometimes confuse it with multiplication.

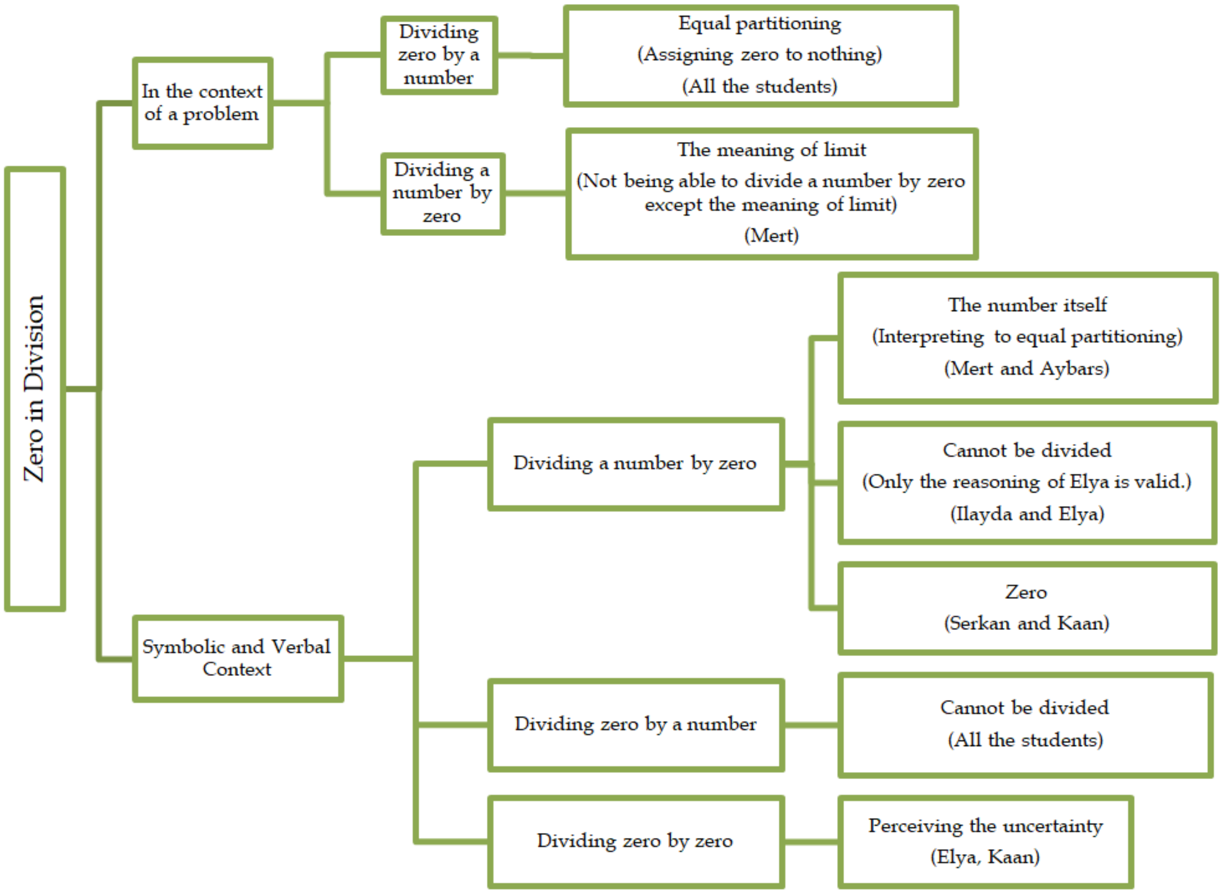
It is thought that this is due to the letter "Ç" [*çıkarma* in Turkish—*subtraction* in English; *çarpma* in Turkish—*multiplication* in English], which is the initial letter of the two words in Turkish. It can be thought that this situation is caused by the verbal-auditory perceptions of the students. Although these insights are very different from those of the students with N-LD, the similarity between the two students with LD may indicate that students have similar insights among themselves.

3.3. Students' Understandings of Zero in Division

In terms of zero in division, students' understandings are as summarized in Figure 6.

Figure 6

Students' understandings of zero in division

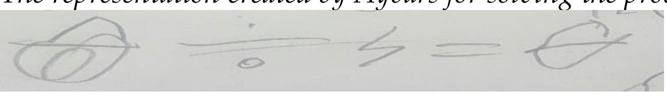


3.3.1. The context of a problem

Dividing zero by a number - equal partitioning. All of the students gave the answer zero to the division of zero by a number asked in the context of a problem by indicating nothing. For example, to the question "You do not have any apples in the basket, if I want to divide them into groups of four, how many apples will there be in each group?" Mert responded as "There are none. It is zero."

Figure 7

The representation created by Aybars for solving the problem



Aybars, as shown in Figure 7, drew a basket and wrote zero in it, saying "If there are no apples in the basket, we will divide it by four", and found the result as zero.

Dividing a number by zero - the limit meaning. In order to examine students' insights on dividing a number by zero, that is to say, dividing by zero with the limit meaning, the problem given to the students is "Seda wants to measure the height of the door of the house below. Seda's ruler is a 0-unit ruler. According to this, what is the height of the door? Let's help Seda to measure the height of the door with a 0-unit ruler." While the division of a number by zero except for the limit meaning is undefined, it is examined whether the infinity would be obtained by dividing a number by zero in the limit meaning by means of this question. Except for Serkan and Mert, all the students first questioned the expression of 0-unit in the question. With the exception of Mert with LD, all students stated that it was not possible to partition a length using a 0-unit ruler.

İ: Seda is measuring the height of the door with a 0-unit ruler. She cannot do it with a 0-unit ruler… This is because it means the ruler has no length. We can, it is zero…this is because zero does not have a length. For instance, if we want to divide this [showing the pen] by the units with no length. No…she can act by rote, but then it will be small or big.
A: It is 0-unit, I cannot help, it is tiny [he puts a dot]. It is 0, I mean, she cannot measure the height of the door, let's say it is 2-meters tall, but we are trying to measure it with a 0-mm or 0-cm unit, no we can't.
E: I don't know, because there is no ruler with a 0-unit. No ruler with 0-unit.

It is consistent that students indicate it is impossible to divide a length by 0-unit with the fact that a number cannot be divided by zero except for the limit meaning. However, it was only Mert who interpreted the question with the limit meaning. Mert stated that an infinite number of pieces could be obtained by associating the partitioning with a 0-unit ruler with partitioning an object into microscopic parts. Mert expressed the division of a specific length by zero in the context of obtaining millions. This corresponds to the fact that division of a specific number by zero in the limit meaning is infinite. This approach by Mert is the same as the mathematical expression $lim_{x \to 0} \frac{a}{x} = \infty$. As long as there is no convergence (partitioning an object into microscopic parts as Mert says), division of a number by zero is not ∞.

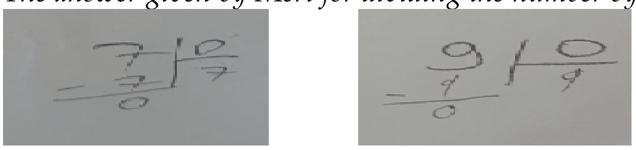
3.3.2. Symbolic context

Dividing a number by zero. Three different situations have emerged as a result of the direct examination of how the students will be able to interpret that the division of a number by zero except for the limit meaning.

a. The number itself

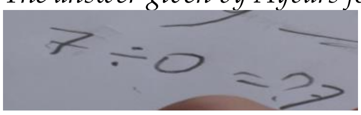
Mert with LD and Aybars with N-LD expressed that the result of the division of a number by zero is the number itself. While Mert correctly interpreted the division of a number by zero in the context of a problem in an intuitive way, he could not interpret it in a symbolic context. Mert answered "When we are asked to divide 7 by 0, I will look for zero in 7. Therefore, 7 is taken down, below zero". Mert was asked to divide 9 by zero (see Figure 8).

Figure 8
The answer given by Mert for dividing the number by zero



Except thinking that zero existed 9 times within 9, Mert did everything (multiplication and subtraction operations in the application of the division algorithm) in his own algorithm (in the multiplication of the result and the dividend, he wrote 9 for the result by perceiving 9 on the top). Similarly, Aybars was also working on the meaning of equal partitioning. However, Aybars perceived the division of a number by zero as making no change (see Figure 9).

Figure 9
The answer given by Aybars for dividing the number by zero



He explained this idea by saying that "7 is divided by 0, the result is 7. There is no change. Let's say there is a basket, there are seven apples in it. We will divide it by zero. Nothing needs to change when dividing by zero". He generalized the division of a number by zero in this way by saying "if 0 is the divider and a natural number bigger than 0 is the dividend, the result will be that number." Therefore, it can be seen that students with N-LD generate algorithms by producing various reasons in the situations they do not interpret.

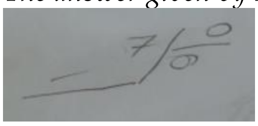
b. Cannot be divided

Ilayda with LD and Elya with N-LD said that a number cannot be divided by zero. Ilayda arrived at the correct answer by saying that "7 is divided by zero. This cannot be done because there is no zero in 7", yet her reasoning is not valid. Not being able to do this operation can only be justified when it is thought of as the opposite of multiplication. Elya reached her decision of not dividing by this justification. Elya first said, "There are 7 zeros in 7," but then she decided that when the dividend was zero, it could not be divided by considering the paradox of multiplying zero by 7: "No, because whatever I multiply by 0, the result is 0." Elya decided that the result would not be 7 as she realized that the dividend was zero whatever the divider was while the result was zero. As can be seen, like Mert and Aybars, Elya firstly stated that the result would be the number itself, but then decided that a number could not be divided by zero by considering the opposite of multiplication.

c. Zero

Kaan with N-LD and Serkan with LD said that dividing the number by zero would be zero again. Serkan arrived at the zero result by interpreting the common algorithm of division. He said for the division of 7 by zero "How many zeros are there in 7? In 7, there are 0 zeros. Kaan stated that the result would be zero even though he started by connecting it with the opposite of multiplication, division.

Figure 10
The answer given by Kaan for dividing the number by zero



> K: If I divide 7 by zero, it's zero, because it doesn't give this when multiplying. Let's say it did not give 7, for example, to find this... Again, the result will be zero. Because it's less than zero, it cannot be -1 or something. If we multiply zero by zero, I multiply nothing by nothing, it will be zero. It doesn't mean anything. It is zero, I mean, when I multiply zero by zero, in no way can it be a number greater than zero. […] Now when 7x0=0, it is zero.

As can be seen, Kaan tried to obtain a result for the division by starting with multiplication, but he applied the algorithm incorrectly. If Kaan had applied this in dividing 0 by 7, he would have got the right result by starting from multiplication. However, when Kaan thought that the result would not be 7 in multiplying zero by zero, he changed the places of the divider, the dividend, and the result in the division, and he shaped his answer in this way by insisting that the result must be 0.

Except for Elya, all the students, who were directly asked about the division of a number by zero, could not interpret that a number cannot be divided by zero. In another way, students' conceptions of the division of a number by zero were examined indirectly with "How many zeros must be summed to have 4?" With this question, the division of a number by zero is examined by asking about the lost multiplier in the multiplication in the context of repetitive collection. The students, except for Serkan with LD, replied that there would be no such multiplier. In the context of addition, when they were asked about the division of 4 by zero, they mentioned unidentifiability and that there was no such number. To this question Aybars answered that we would not have four even if we sum a million zeros. However, Serkan firstly had difficulty in understanding the question:

> S: How can we get 4 when we sum 4 zeros?
> R: How many zeros should we sum to get 4?

S: It is stated here that when we sum 4 zeros, the result is 4. When we add another number to zero, the result is not 4. This is because you are summing, it is written 4 zeros there, it is zero, you have already written 4 there.
R: Hımm, think about that [by adding a zero] how many zeros will result in four or can the sum of zeros result in four?
S: When we have 4 more zeros here (to the blank) maybe we can find 4.
R: Why?
S: This is because it is written here how many zeros can result in 4. Then we should 1 2 3 4 5 6 7 8 ...
R: If I sum two zeros, is the result 4?
S: No. If there are 2 zeros, the result is 2. But it is asked here how many should be summed…
R: What is the sum of zero and zero, Serkan?
S: Zero and zero, zero…
R: What do you think about the question how many zeros will result in 4?
S: ......

As can be seen, Serkan had difficulty in understanding the question and what 4 meant, recognizing what was meant by which number and how many. This may be a problem related to verbal auditory perception. The student was aware that the sum of zero and zero was zero. According to his algorithm, the number on the top must be 4 of addition with zero for the total number to be four. However, the point in the question is the sum of only zeros. Although he could not match this with his own algorithm, he did not express that there would not be a multiplier like that.

In addition to the meaning of repetitive addition, the students were asked to form a direct relationship between the multiplication operation and the division operation with the question "Which number must be multiplied by zero to get four?" For this question, by using their own algorithms, Mert and Serkan stated that the result would not be 4 if zero was on the top, and the result would be 4 when 4 was on the top. Other students stated that the result for the multiplication of a number by zero could not be 4. Aybars indicated that the result would be zero when any number was multiplied by zero.

Dividing Zero by a Number. Students stated that zero could not be divided by a number. Students perceiving division as looking for another number in a number (except Kaan) stated that there could be no number in zero. For example,

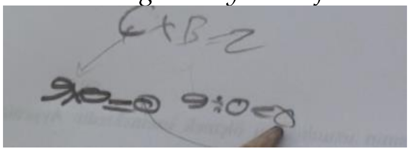
I: (0/9) we cannot divide. It cannot be divided because… there is no nine in zero. (0/247) it cannot be divided either because there is no 247 in it. (0/1) no 1 in it.
R: How many nines are there in zero?
I: None.
R: How else can you express 'none' in math?
I: ... I cannot think of anything. This is because there is no nine in zero. No nine in zero because zero is not even a number.

The reason for the interpretation by Ilayda may be that she did not consider zero as a number. Kaan, who did not assign nothing to zero until the operations, assigned nothing to zero in all the operations with zero. It can be concluded that Kaan's assigning nothing to zero caused him not to interpret the operations with zero and thus to misinterpret the operations.

K: Dividing something by 9... It means this, I mean we cannot divide. If we multiply 9 and 0, the result is 0; in the same vein, if we divide 0 by 9, the result is the same. I guess I got complicated. If we multiply 9 and zero, it is 0. When we divide 9 by zero, the result is zero, I mean, it is the opposite [He writes the things Figure 11].

Figure 11

The answer given by Kaan for dividing 9 by zero



In fact, Kaan, who was aware of the meaning of division as the opposite of multiplication and distinguished the terms dividend, quotient, and divider, could not arrive at the right result since he assigned nothing to zero in the operations with zero:

R: What does it mean, what is the result then?

K: Dividing nothing by zero means dividing nothing by 9. I mean we cannot divide.

R: Alright. Can we divide zero by 747?

K: No, we cannot.

R: Why?

K: Because it is like dividing nothing by 747, I mean, we cannot divide it; it is like dividing nothing by nothing.

For the question that seeks that the other multiplier for the result to be zero when a fixed multiplier is given, that is, the question that examines the division of zero by a number through multiplication, all the students except for Serkan and Mert gave the answer zero. Serkan and Mert used their own algorithms for all the operations with zero. The students were able to respond correctly for the division of zero in the context of multiplication, but they could not do it in the context of division. The fact that zero is not seen as a number, assigning nothing to zero, and the one-directional and limited conceptual understanding of the operations affect the interpretation of the operations with zero.

Dividing Zero by Zero. When the division of zero by zero is questioned symbolically, it is observed that only Elya and Kaan were intuitively aware of the idea of uncertainty. While Serkan and Ilayda with LD and Aybars with N-LD gave the answer zero, Mert said "there is one zero in zero" and gave the answer that the result was one. As Serkan responded by applying his own algorithm, he stated that the result could be all numbers only if the zero was at the top. When asked about the result of dividing zero by zero, Elya first said zero. She then realized that the answer could be all the natural numbers. However, because she did not have the perception of uncertainty and thought that the result should be a single number, she expressed that it must be zero by saying "(for 0/0) The quotient may be 7, or any number, but since there is not a definite number, I say zero." Therefore, the concept of uncertainty can be said to create cognitive conflict. In fact, Elya, who was aware that the result of the multiplication of any number by zero would be zero, thought that the answer of dividing zero by zero would only be zero, not all the numbers, by thinking that the quotient must be a specific number. Kaan considered division as the opposite of multiplication and expressed the idea of uncertainty as infinite. It is important for Kaan and Elya to make the correct interpretations because they did not examine division in terms of the meaning of equal partitioning and they used the meaning of multiplication.

3.4. The Similarities and Differences in the Understandings of the Students with LD and N-LD Regarding Arithmetical Operations with Zero

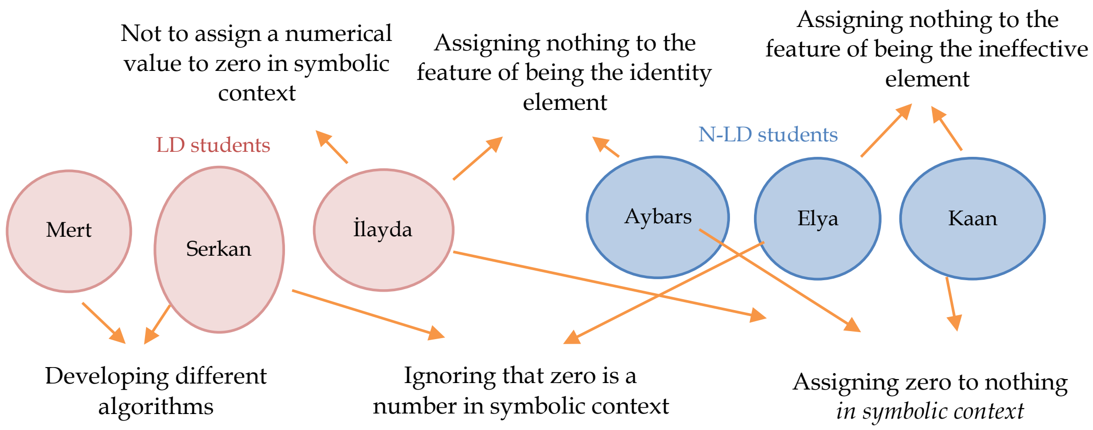
The similarities and differences in the understandings of the students with LD and N-LD can be examined in Table 2 and Figure 12-14, respectively.

Table 2
The similarities in the understandings of students regarding the four operations with zero

Zero in addition and subtraction	*Zero in multiplication*	*Zero in division*
In the context of a problem, they assign zero to nothing. They could not interpret that zero is an identity element.	In the context of a problem, they assign zero to nothing. They could not make sense of the feature of zero as the absorbing element.	In the context of a problem, the division of zero by a number, they assign zero to nothing (equal partitioning). In the symbolic context the division of zero by a number their answer is zero.

As a result of the findings in the context of a problem in addition and subtraction, multiplication, and division of zero by a number all students including Mert and Serkan, who had significant misconceptions in the symbolic application of the basic arithmetic operations, responded correctly to the daily life problems requiring the operations. Moreover, in the symbolic context of the division of zero by a number the interpretations of the students depended on the division algorithm. The students except for Kaan with N-LD thought that "there could be no number in zero" and so their answer was zero. Kaan was aware of the division algorithm process but he assigned nothing to zero. Consequently, his answer was zero. In addition, no student was able to make sense of the fact that zero is an identity element; also in multiplication no student was able to make sense of the fact that zero is an absorbing element.

Figure 12
The differences in the students' understandings regarding zero in addition and subtraction



It is noteworthy that in the addition, subtraction, and multiplication Mert and Serkan with LD had different understandings and different algorithms that they had constructed by ignoring the commutative property of addition (see Figure 12 and Figure 13). It is also stated in the literature that students with LD are not aware of the commutative property of addition (e.g., Gersten et al., 2005; Gersten et al., 2008). They assigned their knowledge for the addition and subtraction with other numbers to zero and had different approaches to addition and subtraction with zero. Kaan and Aybars with N-LD and Ilayda with LD assigned zero to nothing in the addition and subtraction operations (see Figure 12). It is noteworthy that among the students who did not see zero as a number and thus assigned nothing to zero, Elya with N-LD and Serkan with LD had the same insights in the symbolic context (see Figure 12). In addition, Ilayda's assigning nothing to zero in the symbolic context attributed this to not to assigning a value to zero.

Figure 13

The differences in the students' understandings regarding zero in multiplication

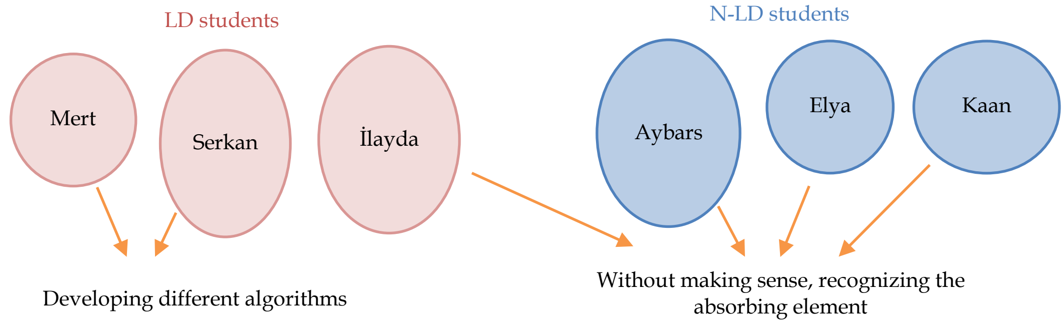
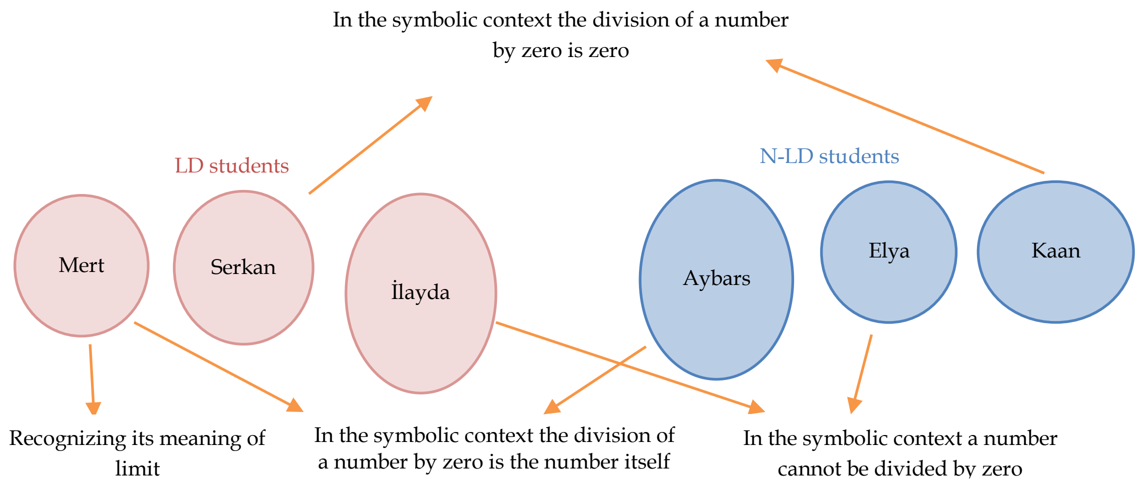


Figure 14

The differences in the students' understandings regarding zero in division



In the context of the problem, the students arrived at the result zero by assigning zero to nothing in the division of zero by a number (see Table 2). That is, they assigned zero to something that does not exist by interpreting the equal partitioning meaning of division, and found zero by obtaining nothing as a result of partitioning something that does not exist. The students were aware that the number cannot be divided by zero. However, in terms of limit, they, except Mert, could not comment on the division of a number by zero and its results (see Figure 14). This may be due to the fact that the students do not find the 0-unit ruler meaningful. Nevertheless, Mert associated partitioning the height of the door with a zero-unit ruler by obtaining microscopic fragments by considering physics and biology. Thus, he could evaluate the result by perceiving the division of a number by zero with its limit meaning. This interpretation is noteworthy for a student with LD.

Three different situations have emerged as a result of a direct examination of how the students will be able to interpret that the division of a number by zero is undefined except for the limit meaning. Each case was observed for two students with LD and N-LD. However, although the results achieved by the students were the same, their justifications varied. Mert with LD, who concluded that the division of a number by zero results in the number itself, justified this result by considering that zero exists in a number as much as the number, while Aybars with N-LD indicated that zero had no meaning in the division of a number by zero by assigning nothing to zero and the result would be that number. He could not comment on the division of a number by zero in the symbolic context as he did in the context of the problem. The students assigned nothing to zero in the division operation, but they also differed in their interpretation and achieved different results. Although both Aybars and Kaan assigned nothing to zero in the division operation, the results they obtained were different. Only Elya and Ilayda indicated that the ordinary algorithm could not be applied to the fact that a number could not be divided by zero. However, it was only Elya with N-LD who could justify it with the inversion of the multiplication operation. Students who could not answer (except Elya) when they were asked to divide a number by zero could not make sense of the fact that the number could not be divided by zero, while students other than Serkan with LD replied that there could not be such a multiplier when investigating the division of a number by zero by questioning the lost multiplier in the context of repetitive addition. When asked by associating with the multiplication operation, the students, except for Mert and Serkan, concluded that such a process could not be carried out. As Mert and Serkan used their own algorithms, it is not surprising that they arrived at these answers. Because the students' algorithms were different, this question did not create perceptions the same as those of the other students in the symbolic context for them. Because Mert and Serkan with LD had their own different understanding and algorithms, the questions asked according to the ordinary algorithm were meaningless for the students in some cases. However, in particular, Serkan is consistent in his own logic and in his responses within his algorithm. Among all participants, the most consistent student is Serkan. Although his algorithm is contrary to the known one and in a rote fashion, it is noteworthy that Serkan always shows consistency.

The division operations in which zero is involved do not contain a satisfactory arithmetical response allowing the possibility of dealing with an abstract mathematical idea that is accessible to everyone independent of the mathematical background and questionable within a reasonable time (Crespo & Nicol, 2006). In this context, the understandings of students with LD and N-LD are investigated with the same questions with the idea that the students' perceptions of zero can be evaluated independently of their mathematical backgrounds in the clinical interviews. The division of zero by zero, of a number by zero, and of zero by a number, that is, zero as the quotient, dividend, or divider, are all examined. In the division operations where zero is included, the basic categories are referring to division as the opposite of multiplication or to equal partitioning, interpreting zero as nothing, the missing multiplier, or the number of dividers in the dividend in the division algorithm. Students' understanding of the operations in the symbolic context often varies between the LD and N-LD student groups. There is a heterogeneous distribution of students' understandings of the division by zero due to both the reflections of their conception of the nature of zero and the difficulty of division by zero. When division involving zero was asked about in the context of repetitive addition and multiplication, all the students, except for Serkan and Mert with LD, gave the correct answer. This situation is apparently caused by the different algorithms developed by Serkan and Mert regarding addition and multiplication. In this context, while all the students except for Serkan and Mert were aware that division of a number by zero is undefined except for the limit meaning, only Mert obtained infinity in terms of limit.

4. Discussion, Conclusion and Recommendations

The current study investigates the similarities and differences between the understandings of students with LD and N-LD of zero in operations. In basic arithmetic operations, zero resulted in very different insights from other numbers. This was true for all students with LD in all four operations, but was more apparent in the multiplication and division operations for students with N-LD. It is observed that the fact that zero is not seen as a number, nothing is assigned to zero, and the conceptual understandings regarding the operations are generally one-way or limited (not being aware of the interrelation with operations) affects and even complicates the interpretation of operations with zero. It is already indicated in the literature that the most common misconception is related to zero not being a number and assigning nothing to zero (e.g., Reys & Grouws, 1975; Russell & Chernoff, 2011). It appears that the students' limitation of reasoning with the algorithms they have learned and their overgeneralization of these features of the algorithms limit their interpretation of the place of zero in operations. Therefore, due to the nature of zero, students' understandings of the assignment of nothing to zero and of zero to nothing are reflected in their perceptions about the division operation. In the present study, which concluded that students' understandings about zero are different from those about other numbers, and even some students do not see it as a number, all the students with LD and N-LD had difficulty in all the operations with zero. Similarly, Grouws and Reys (1975) also stated in their study with 4th, 6th, and 7th grade students that all division operations involving zero were difficult for students.

It is seen that the students could not interpret the limit meaning of zero in the division operation even in the context of the problem, but they could understand that it is impossible to divide a number by zero other than with the limit meaning. It appears that the students misinterpreted the division process involving zero because they were searching for the number of the dividers in the dividend in a division operation. In the same vein, related studies also found that students, teacher candidates, and even teachers mostly looked for the number of dividers in the dividend in division operations with zero (Ball, 1990; Reys & Grouws, 1975; Tirosh, 2002; Wheeler & Feghali, 1983). Tsamir and Tirosh (2002) provided an effective explanation for this:

"Division by zero is usually the first undefined mathematical operation that students encounter during their school studies. Clearly, the mere existence of an undefined mathematical term violates the intuitive, numeric-answer belief. Adherence to this belief might result in assigning numerical values to expressions involving division by zero." (Tsamir & Tirosh, 2002, p. 332)

Students were able to intuitively access the concepts of unidentifiability and uncertainty when they addressed the division operation as the opposite of the multiplication operation, not only in terms of equal partitioning. As aforementioned, this is an example of the fact that zero is not seen as a number and the conceptual understanding of the operations is one-way and limited. The limiting of students' thinking by algorithms and their overgeneralization of the features related to these algorithms limit their interpretation of the place of zero in operations. Therefore, we can see the importance of employing operations with various meanings and interpretations in teaching (e.g., repetitive addition, lost multiplier, in the context of a problem context, and a symbol) and emphasizing the relationships between the operations.

In the statements they made about the role of zero in multiplication, the students only used words such as "swallowing" and "eating". None of the participants could conceptually explain why zero is the absorbing element based on the repetitive addition interpretation of multiplication. It is thought that this expression causes the student to construct different insights regarding the multiplication operation instead of focusing on its features. In fact, the name given to this feature is thought not to reflect the conceptual meaning of the feature. This name may confuse children. It is suggested that the role of zero in multiplication needs to be explained with justifications. For this reason, the mathematical meaning of multiplication by zero should be presented first and then this feature should be introduced.

Considering the reasons the LD students developed different algorithms, it was seen that there were times when the students were unable to make sense of a sentence as a whole (Namkung & Peng, 2018; Pierangelo & Giuliani, 2006), and there were differences in the verbal auditory perceptions of students (for instance, to the question "if I have no eggs in four baskets, how many eggs do I have?", giving the answer "4 plus 4 is 8", or perceiving the expression 0-unit as zero one). Therefore, the different interpretations of these students due to their problems with verbal auditory perceptions may cause different perceptions, and the search for logic of the mind may be directed towards developing a consistent algorithm. In the same vein, Hunt and Empson (2014) stated that students with LD can develop their own strategies to understand fractions. The possible reasons why students interpret zero in terms of its place at the top or bottom in this way in their wrong algorithms of the operations may be due to the expressions such as "For 20-3, we cannot subtract three from zero; therefore, we take ten from here". In line with this, the students confused multiplication and subtraction as terms. Therefore, as it should be considered that students with

LD can even misinterpret the correct words, it can be useful for teachers to pay more attention to the words they choose when teaching these students and not to use misleading language.

The present study shows that although understandings of LD students are beyond the known ones, they may have similar ways of thinking. It is noteworthy that various algorithms of LD students regarding the operations of addition, subtraction, and multiplication are only valid for operations with zero. With other numbers, they perform the operations correctly up to a certain place. Similarly, Clements and Sarama (2009) state that children think in different ways about zero and create special rules for this exceptional number. Therefore, our study examining students' perceptions of the concept of zero has led to the emergence of these different algorithms of the students with LD. This could also be an example of the different nature of zero and different perceptions of zero from those of other numbers. In this context, it is important to consider teaching zero with the conceptual reasons underlying the operations by emphasizing that zero is a number, the commutative property of the addition operation, and the repetitive addition meaning of the multiplication operation.

In the present study, with an in-depth examination, the existence of different algorithms developed by LD students for addition, subtraction, and multiplication was determined. The generalizability of this finding can be examined by a study with a large group of students. In this way, it is thought that ideas and clues can be obtained about the mathematical thinking systems and perceptions of LD students, and why these students, who are known to have problems with the four operations, have problems at this point. In addition, based on the findings about division by zero in the study, it is thought that it is important to examine the reasoning of the challenging concepts of mathematics such as infinity and limit in accordance with the level of the student, in terms of showing the boundaries of the mathematical thinking of LD students.

Author contributions: Dilşad Güven Akdeniz: Data Collection, Data Analysis, Conceptualization, Methodology, Writing- original draft, Writing – reviewing, Editing. Esra Selcen Yakıcı Topbaş: Data Collection, Data Analysis, Conceptualization, Methodology, Editing. Ziya Argün: Supervision. All authors agreed with the results and conclusions.

Funding: No funding source is reported for this study.

Declaration of interest: No conflict of interest is declared by authors.

References

American Psychiatric Association. (2013). *Diagnostic and statical manual of mental disorders: DSM-5.* American Psychiatric Association.

Andersson, U., & Östergren, R. (2012). Number magnitude processing and basic cognitive functions in children with mathematical learning disabilities. *Learning and Individual Differences*, *22*(6), 701-714. https://doi.org/10.1016/j.lindif.2012.05.004

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education, 21*(2), 132-144. https://doi.org/10.5951/jresematheduc.21.2.0132

Blake, R., & Verhille, C. (1985). The story of 0. *For the Learning of Mathematics, 5*(3), 35-47.

Boyer, C. B., & Merzbach, U. C. (2011). *A history of mathematics.* John Wiley & Sons.

Catterall, R. (2006). *Exploring children's conception of zero* [Doctoral dissertation]. Sheffield Hallam University.

Clements, D. H., & Sarama, J. (2009). *Learning and teaching early math: The learning trajectories approach.* Routledge.

Council for Exceptional Children. (n.d.). *Individuals with Disabilities Education Act 2004.* Retrieved from https://sites.ed.gov/idea/building-the-legacy-idea-2004/#bs_tab150290691146380,citem_3dad-0f9e

Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics, 106*(2), 84-97. https://doi.org/10.1111/j.1949-8594.2006.tb18138.x

Duncan, H. F. (1971). Division by Zero. *Arithmetic Teacher, 18*(6), 381-382.

Gallardo, A., & Hernández, A. (2005). The duality of zero in the transition from arithmetic to algebra. *International Group for the Psychology of Mathematics Education, 3*, 17-24.

Gallardo, A., & Hernández, A. (2006). The zero and negativity among secondary school students. In J. Novotna, H. Moraova, M. Kratka, & N. Stehlikova (Eds.), *Proceedings of the 30 th Conference of the International Group for the Psychology of Mathematics Education* (pp. 153-160). PME.

Geary, D. C. (1990). A componential analysis of an early learning deficit in mathematics. *Journal of Experimental Child Psychology*, *49*(3), 363-383. https://doi.org/10.1016/0022-0965(90)90065-G

Gersten, R., Chard, D. J., Jayanthi, M., Baker, S. K., Morphy, P., & Flojo, J. (2008). Mathematics instruction for students with learning disabilities or difficulty learning mathematics: A synthesis of the intervention research. Retrieved from *Center on Instruction*. http://www.centeroninstruction.org/files/Teaching%20Math%20to%20SLD%20Meta-analysis.pdf

Gersten, R., Jordan, N. C., & Flojo, J. R. (2005). Early identification and interventions for students with mathematics difficulties. *Journal of Learning Disabilities*, *38*(4), 293-304. https://doi.org/10.1177/00222194050380040301

Goldman, S. R., Pellegrino, J. W., & Mertz, D. L. (1988). Extended practice of basic addition facts: Strategy changes in learning-disabled students. *Cognition and Instruction*, *5*(3), 223-265. https://doi.org/10.1207/s1532690xci0503_2

Grouws, D. A., & Reys, R. E. (1975). Division involving zero: An experimental study and its implications. *Arithmetic Teacher, 22*(1), 74-80.

Gullberg, J. (1997). *Mathematics: from the birth of numbers*. Norton & Company.

Güven Akdeniz, D., & Argün, Z. (2021). Learning trajectory of a student with learning disabilities for the concept of length: A teaching experiment. *The Journal of Mathematical Behavior, 64*, 100915. https://doi.org/10.1016/j.jmathb.2021.100915

Güven, N., & Argün, Z. (2018). Width, length, and height conceptions of students with learning disabilities. *Issues in Educational Research, 28*(1), 77-98. https://doi/10.3316/437830316660646

Hunt, J. H., & Empson, S. B. (2015). Exploratory study of informal strategies for equal sharing problems of students with learning disabilities. *Learning Disability Quarterly, 38*(4), 208-220. https://doi.org/10.1177/0731948714551418

Hunting, R. P. (1997). Clinical interview methods in mathematics education research and practice. *The Journal of Mathematical Behavior, 16*(2), 145-165. https://doi.org/10.1016/S0732-3123(97)90023-7

Inhelder, B., & Piaget, J. (1969). *The psychology of the child.* Basic Books.

Joseph, G. G. (2008). A brief history of zero. *Iranian Journal for the History of Science, 6*, 37-48

Judge, S., & S. M. Watson (2011). Longitudinal outcomes for mathematics achievement for students with learning disabilities. *The Journal of Educational Research 104*(3), 147-157. https://doi.org/10.1080/00220671003636729

Levenson, E., Tsamir, P., & Tirosh, D. (2007). Neither even nor odd: Sixth grade students' dilemmas regarding the parity of zero. *The Journal of Mathematical Behavior, 26*(2), 83-95. https://doi.org/10.1016/j.jmathb.2007.05.004

Lewis, K. E. (2014). Difference not deficit: Reconceptualizing mathematical learning disabilities. *Journal for Research in Mathematics Education, 45*(3), 351-396. https://doi.org/10.5951/jresematheduc.45.3.0351

Mejias, S., Mussolin, C., Rousselle, L., Grégoire, J., & Noël, M. P. (2012). Numerical and nonnumerical estimation in children with and without mathematical learning disabilities. *Child Neuropsychology*, *18*(6), 550-575. https://doi.org/10.1080/09297049.2011.625355

Merritt, D. J., & Brannon, E. M. (2013). Nothing to it: Precursors to a zero concept in preschoolers. *Behavioural Processes, 93*, 91-97. https://doi.org/10.1016/j.beproc.2012.11.001

Miles, M. B., & Huberman, A. M. (1994). *An expanded sourcebook qualitative data analysis* (2nd. ed.). Sage.

Ministry of National Education (MoNE). (2006). *Özel eğitim hizmetleri yönetmeliği*. [Special education services regulation] Retrieved from https://orgm.meb.gov.tr/meb_iys_dosyalar/2006_10/10111226_ozel_egitim_hizmetleri_yonetmeligi_son.pdf

Ministry of National Education (MoNE). (2018). *Matematik dersi öğretim programı (İlkokul ve ortaokul 1, 2, 3, 4, 5, 6, 7 ve 8. sınıflar)* [Mathematics Teaching Program (1st-8th grades)]. Author.

Namkung, J., & Peng, P. (2018). Learning disabilities. In B. B. Frey (Ed.), *The Sage Encyclopedia of Educational Research, Measurement, and Evaluation* (pp. 243-278). Sage.

Nath, R. (2012). A brief history of zero. *Dream 2047, 14*(8), 38-39.

Piazza, M., Facoetti, A. Trussardi, A. N., Berteletti, I., Conte, S., Lucangeli, D., … Zorzi, M. (2010). Developmental trajectory of number acuity reveals a severe impairment in developmental dyscalculia. *Cognition, 116*, 33–41. https://doi.org/10.1016/j.cognition.2010.03.012

Pierangelo, R., & Giuliani, G. A. (2006). *Learning disabilities: A practical approach to foundations, assessment, diagnosis, and teaching*. Pearson College Division.

Quinn, R. J., Lamberg, T. D., & Perrin, J. R. (2008). Teacher perceptions of division by zero. *The Clearing House: A Journal of Educational Strategies, Issues and Ideas, 81*(3), 101-104. https://doi.org/10.3200/TCHS.81.3.101-104

Reys, R. E. (1974). Division by zero: An area of needed research. *Arithmetic Teacher, 21*, 153-157.

Reys, R. E., & Grouws, D. A. (1975). Division involving zero: Some revealing thoughts from interviewing children. *School Science and Mathematics, 75*(7), 593-605. https://doi.org/10.1111/j.1949-8594.1975.tb09861.x

Russell, G., & Chernoff, E. J. (2011). Seeking more than nothing: Two elementary teachers' conceptions of zero. *The Mathematics Enthusiast, 8*(1), 77-112.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children.* Routledge.

Schoenfeld, A. (1994). Reflections on doing and teaching mathematics. In A. Schoenfeld (Ed.), *Mathematical Thinking and Problem Solving* (pp. 53-72). Lawrence Erlbaum Associates.

Seife, C. (2000). *Zero: The biography of a dangerous idea*. Penguin.

Sen, S. K., & Agarwal, R. P. (2015). *Zero: a landmark discovery, the dreadful void, and the ultimate mind*. Academic Press.

Siegler, R. S., & Shrager, J. (1984). Strategy choice in addition and subtraction: How do children *know* what to do? In C. Sophian (Ed.), *Origins of Cognitive Skills* (pp. 229–293). Erlbaum.

Stake, R. (1995). *The art of case study research*. Sage.

Tsamir, P., & Sheffer, R. (2000). Concrete and formal arguments: The case of division by zero. *Mathematics Education Research Journal, 12*(2), 92-106. https://doi.org/10.1007/BF03217078

Tsamir, P., & Tirosh, D. (2002). Intuitive beliefs, formal definitions and undefined operations: Cases of division by zero. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: A hidden variable in mathematics education?* (pp. 331-344). Springer.

Tsamir, P., Sheffer, R., & Tirosh, D. (2000). Intuitions and undefined operations: The cases of division by zero. *Focus on Learning Problems in Mathematics, 22*(1), 1-16.

Watson, J. M. (1991). Models to show the ımpossibility of division by zero. *School Science and Mathematics, 91*(8), 373-76. https://doi.org/10.1111/j.1949-8594.1991.tb12123.x

Wellman, H. M., & Miller, K. F. (1986). Thinking about nothing: Development of concepts of zero. *British Journal of Developmental Psychology, 4*(1), 31-42. https://doi.org/10.1111/j.2044-835X.1986.tb00995.x

Wheeler, M. M., & Feghali, I. (1983). Much ado about nothing: Preservice elementary school teachers' concept of zero. *Journal for Research in Mathematics Education, 14*(3), 147-155. https://doi.org/10.2307/748378

Yin, R. K. (2013). *Case study research: Design and methods*. Sage.

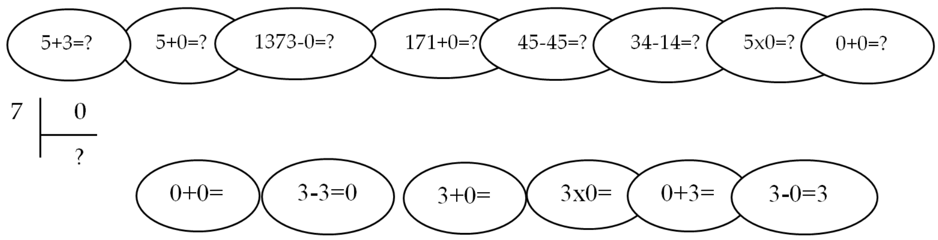
Appendix 1. Clinical Interview Form

1) I have 3 biscuits. How many biscuits will I have left if I eat them all? Number Meaning (Arithmetic function)

2) I have 3 biscuits. I pretended to eat some of them, but I didn't eat any. How many biscuits do I have left? Number Meaning (Arithmetic function)

3) I have 3 packages. There are no eggs in them. How many eggs do I have then? Number Meaning (Arithmetic function)

4) Examine the following procedures. Number Meaning (Arithmetic function)



Why were the results like this? What does zero say here? What is the function of zero?

5) $5 + 3 =?$ $5 + 0 =?$ $1373 - 0 =?$ $171 + 0 =?$ $45 - 45 =?$ $34 - 14 =?$ $5x0 =?$ (Addition would increase the result, why did it not in this case? Subtraction would decrease the result, why did it not in this case? Why did it become zero when you multiplied 5 by zero?)

6) What number do I add to 2 to get 0?

7) What number do I add to 10 if the result will be 0?

8) $0x12 =?$ $0x135 =?$ $12x0 =?$ $145x0 =?$

9) $\frac{0}{9} =?$ $0/247 =?$ $0/1 =?$

10) There are no eggs in a basket. If I divide these eggs into 4 groups, how many eggs will there be in each group? (Number meaning)

11) Seda wants to measure the height of the door of the house below. Seda's ruler is a 0-unit ruler. What is the height of the door?

Let's help Seda measure the height of the door with a 0-unit ruler.