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Research Article

To what extent can teachers and preservice teachers predict students' mathematical thinking? A qualitative analysis

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In studies over recent years, there has been an increasing interest in teachers' predicting middle school students' thinking processes. However, as far as we are aware, there are no studies examining students' thinking in terms of mathematical thinking components. This study primarily aimed to determine the mathematical thinking of middle school students. Therefore, the study examined how six mathematics teachers and 24 preservice mathematics teachers (from first to fourth grade) predicted the mathematical thinking of 96 middle school students. In this context, the predictions were categorized according to the sub-components of mathematical thinking: conjecturing, specializing, justifying and convincing, and generalizing. Regarding the *conjecturing,* the teachers explained students' prediction of their mathematical thinking in more detail than preservice teachers. Regarding the *specializing*, the study, both groups of teachers could not predict that the students could express different situations in their problem solutions. Within the scope of the *justifying and convincing*, the preservice teachers had different perspectives on problem solving compared to the teachers. In regard to the *generalizing*, teachers and preservice teachers made similar predictions but all groups from first to fourth grade lack experience for this component. It can be stated that preservice teachers' interaction with more students will be effective in predicting students' mathematical thinking. The same is true for teachers, as it is believed that greater experience will be beneficial.

Keywords: Mathematics teachers; Middle school students; Preservice teachers; Teacher predicting

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1. Introduction

There is an expectation that teachers will have knowledge and awareness of their students' mathematical learning, in addition to being accepted in educational circles as a necessary aspect of teaching (Even & Tirosh, 2008). One of the basic principles of mathematics teaching is to create an education based on students' understanding and thinking about mathematics (Ball, 2001; National Research Council [NRC], 2005). The National Council of Mathematics Teachers [NCTM] (2014) has defined one component of effective teaching as "eliciting evidence of students' current mathematical understanding and using it as the basis for making instructional decisions" (p. 53). Teachers play a critical role in achieving this. The more information teachers have about students'

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thoughts, the more options they can offer for success (Darling-Hammond, 1994). Hughes (2006) states that to teach mathematics effectively, teachers should know how students learn and how they think. Studies have also emphasized the importance of understanding students' thoughts and making changes in the education process accordingly (Jacobs et al., 2010; Lee & Cross Francis, 2018; Sherin et al., 2011).

Considering that mathematical thinking is about understanding existing ideas, discovering the relationships between ideas, and expressing the foundations of relationships (Lutfiyya, 1998), the development and evaluation of students' mathematical thinking hold a key position. Trying to understand students' mathematical thinking styles not only ensures that lessons are efficient, but also improves the mathematical knowledge and learning of teachers (McLeman & Cavell, 2009). It is therefore crucial to understand students' mathematical thinking in order to take appropriate steps in the teaching process. Evaluation of students' mathematical thinking in light of teachers' predictions will result in more effective teaching. By evaluating students' mathematical thinking in the context of teachers' predictions, more effective practices will be developed. It will be more effective to evaluate every stage of teaching by examining the predictions made by preservice teachers as well as teachers.

1.1. Mathematical Thinking

Mathematical thinking can be thought of as the direct or indirect use of mathematical knowledge, concepts, and processes in solving problems and helping to solve problems in any subject (Henderson et al., 2001). In this context, many researchers have tried to define mathematical thinking in various ways (Keith, 2000; Liu, 2003; Mason et al., 2010; Polya, 1973; Schoenfeld, 1994). Polya (1973) defines it as the process of researching events, experimenting with the results, making predictions, forming and testing hypotheses, and collecting and analyzing data. Mason et al. (2010), on the other hand, express mathematical thinking as a dynamic process that facilitates understanding complex structures by combining thoughts. Mathematical thinking can be developed by solving problems carefully, making connections between thinking and actions by transferring what has been gained to experience, working on the problem solving process, and understanding the relationship of mathematics to real life (Keith, 2000).

Mathematical thinking involves considering and examining a problem through various dimensions rather than simply finding the answer to a problem (Borromeo Ferri, 2003). In this context, mathematical thinking is explained by dividing it into various components. The components of mathematical thinking are determined by aiming at reaching the essence of knowledge, having a mathematical perspective, using problem solving strategies, using one's knowledge effectively, and dealing with mathematical activities (Schoenfeld, 1992). On the other hand, Mason et al. (2010) state that mathematical thinking includes the components of conjecturing, specializing, justifying and convincing, and generalizing.

Specializing is to select systematic examples to understand a problem situation and examine these selected examples of the problem (Burton, 1984). In the problem solving process, finding special cases by giving random values is useful for determining the accuracy of a given situation, while giving a systematic value is efficient in seeing the relationship between the data (Mason et al., 2010).

Generalization can be expressed in the form of individuals reaching more general and comprehensive information based on the relationships between the data from a few examples (Mason et al., 2010). Students' generalization skills begin to develop when a problem' solution is used to solve another problem. Therefore, generalization is not an easy process. In this process, students need to express the relationships between variables mathematically (Driscoll, 2007).

Conjecturing is explained as the process of concluding existing relations by examining the necessary examples and discovering the relationships between the examples before making a certain judgment (Burton, 1984). Identifying and evaluating conjecturing is the mainstay of mathematical thinking. Therefore, conjecturing is considered a cyclical process (Mason et al., 2010).

The justifying and convincing component is based on investigating the reason for a statement being defended and understanding the reasons for the validity of the assumption (Mason et al., 2010). This situation improves students' mathematical thinking, enables them to better understand the concepts, and makes the results they find reasonable (Hersh, 1993).

1.2. Predicting Students' Mathematical Thinking

Teachers' knowledge of students' thinking has a significant impact on classroom teaching and student learning (Ball, 1997; Cai & Jiang, 2017; Hill et al., 2008). Recognition of students' mathematical thinking plays an important role in making instructional decisions, especially when teachers need to respond to students' verbal or written explanations about their mathematical work (Jacobs et al., 2011). For quality teaching, teachers need to know the way their students think as well as learning about alternative pedagogical approaches (Hiebert & Stigler, 2000). Teachers' noticing students' mathematical thinking depends on their teaching and diagnosis competencies (Borromeo Ferri & Blum, 2010). Studies have also emphasized that the more information teachers have about students' thoughts, the more the quality of teaching increases (Ball, 1997; Ball et al., 2008; Fennema et al., 1996). There is also evidence that teachers teach better in parallel with their knowledge of students' thoughts, and students achieve higher success (Carpenter et al., 1989; Fennema et al., 1993).

Jacobs et al. (2010) state that teachers' professional noticing includes attending to students' strategies, interpreting students' understanding, and deciding how to respond based on students' understanding. Kaiser et al. (2015), on the other hand, state that teachers notice in three ways: (a) Perceiving certain events in an instructional environment, (b) Interpreting perceived activities in the classroom, (c) Making decisions, predicting a response to students' activities, or suggesting alternative teaching strategies. This study examines student activities and investigates teachers' and preservice teachers' prediction strategies for these activities.

Various studies have been conducted on teachers' mathematical thinking of students. Nathan and Koedinger (2000) investigated teachers' prediction of students' algebraic development. As a result of the study, they found that teachers underestimated the difficulties experienced by students in symbolic reasoning. In addition, they determined that they considered verbal problems to be more difficult than other problems. In his study, Cai (2005) asked Chinese and US teachers to estimate the difficulty levels of five problems regarding the arithmetic mean. They concluded that US teachers were more predictive to guess-and-check strategies than Chinese teachers. For algebraic strategies, Chinese teachers had a higher level of prediction. Xu et al. (2020) investigated teachers' predictions about students' problem posing. They found inconsistencies between the problems created by the students and the predictions of the teachers. In this context, the researchers concluded that teachers made predictions at a higher level than the problems posed by students for mathematical thinking. Asquith et al. (2007) focused on teachers' knowledge of students' understanding of basic algebraic concepts. In this context, they examined secondary school mathematics teachers' students' understanding of the concept of equal sign and variable. For this purpose, they investigated the level of teachers' prediction of students' understanding of this concept and concluded that there was a great deal of agreement between the student's answers and the teachers' predictions. Helmke and Schrader (1987) investigated teachers' prediction of student achievement. In addition to paying attention to students' performance, teachers with a high level of prediction of students' performance provided hints and individual support for the lesson. This means that teachers are more likely to be successful when predicting students' mathematical thinking in the classroom.

Several studies have also been carried out on preservice teachers' predictions of students' mathematical thinking. Norton et al. (2011) designed an assessment tool to evaluate the effectiveness of a teaching practice course. This tool was designed for preservice teachers to make video-based prediction assessments while analyzing students' mathematical thinking tasks. The preservice teachers analyzed the mathematical models created by the students and used these models to predict how the students would respond to the next task. They concluded that the predictions made with the measurement tool they prepared enabled the preservice teachers to effectively develop and assess their pedagogical content knowledge. Simpson et al.'s (2018) found that the teaching practice course alone did not make a difference to preservice mathematics teachers' predictions of students' mathematical thinking. In addition, the literature was looked at from a different perspective by focusing on the components of mathematical thinking (Mason et al., 2010) separately.

In studies conducted with teachers and preservice teachers, Jacobs et al. (2010) worked with preservice teachers and teachers from four different education levels. The study not only provided participants with teaching experience but also made them notice student thinking based on learning activities. Wilson et al. (2013) investigated how teachers and preservice teachers use learning instructions to make sense of what students think about a basic rational number.

1.3. Importance of the Study

Studies in the field of mathematics education have emphasized that teachers' awareness of students' thought is important in terms of giving effective feedback to students (Carpenter et al., 2000; Thiede et al., 2015). In this context, there is a need for studies that focus on how teachers predict students' mathematical thinking (LaRochelle et al., 2019). Some studies were conducted with teachers (Amador et al., 2022; Asquith et al., 2007; Blömeke et al., 2022; Cai, 2005; Helmke & Schrader, 1987; Nathan & Koedinger, 2000; Ready & Wright, 2011; Thiede et al., 2015; Xu et al., 2020), and some only with preservice teachers (Dick et al., 2020; Norton et al., 2011; Simpson et al., 2018; Star et al., 2011; Teuscher et al., 2017; Tekin-Sitrava et al., 2021; Vacc & Bright, 1999). There are few studies evaluating teachers and preservice teachers together (Cai et al., 2022; Huang & Li, 2012; Jacobs et al., 2010; Wilson et al., 2013; Yang et al., 2021). In this context, the main purpose of the study is to determine how mathematics teachers and preservice teachers predict middle school students' solutions to mathematical thinking tasks.

This study aimed to evaluate the teachers and preservice teachers together regarding predicting the mathematical thinking of middle school students. Teachers' knowledge of students' thinking styles is highly effective in classroom education and student learning (Cai et al., 2018; Fennema & Franke, 1992; Gardner, 1999). At the same time, it is thought that analyzing students' mathematical thinking will be very useful for teachers in making appropriate decisions and developing activities in lessons (Crespo, 2000).

This study aims to contribute to the literature by examining teachers' and preservice teachers' predicting of students' mathematical thinking. To examine participants' mathematical thinking, the study is based on the mathematical thinking components determined by Mason et al. (2010). With a mathematical thinking task prepared for each component, the study attempts to determine teachers and preservice teachers' approaches to predicting students' mathematical thinking. The main question of the research is as follows: How do mathematics teachers and preservice mathematics teachers predict middle school students' responses on mathematical thinking tasks? The following sub-questions were examined within the scope of this main research question.

RQ 1) Which mathematical thinking strategies do middle school students use when faced with a mathematical problem?

RQ 2) How do mathematics teachers solve mathematical thinking tasks?

RQ 3) How do mathematics preservice mathematics teachers solve mathematical thinking tasks?

RQ 4) How do mathematics teachers predict the mathematical thinking of middle school students?

RQ 5) How do preservice mathematics teachers predict the mathematical thinking of middle school students?

RQ 6) What are the differences and similarities between mathematics teachers and preservice mathematics teachers' predicting of students' mathematical thinking?

2. Method

2.1. Research Design

Use of qualitative research methods, which allow direct quotations from participants' statements, explore the nature of the issue in greater depth, and enrich the findings with various examples, has been considered to me more appropriate in line with the aim of this study. Hence, the study adopted a case study design. In the study, more than one layer was examined, as there were middle school students, preservice teachers, and teachers. In this context, the holistic multiple case method, which is a sub-method of case studies, was preferred. In this way, the aim was to collect systematic, comprehensive, and in-depth information about situations in case studies (Patton, 2002).

2.2. Participants

The participants of this study were middle school 7th and 8th-grade (14-15 years old) students, preservice mathematics teachers (1st, 2nd, 3rd, and 4th grades), and mathematics teachers. Middle school students were selected from schools located in regions with different socio-economic levels. Students were chosen to provide a maximum diversity according to gender and grade level from six different schools (a central school, a private school, a village school affiliated to the center, a district school, a town school, and a village school). By selecting 16 students from each middle school, 96 middle school students participated in the study. Eight students per school were female, and eight were male. As well, eight of the students were in 7th grade, and eight were in 8th grade.

Furthermore, 24 preservice mathematics teachers studying in the department of elementary mathematics teaching were also selected as participants. A total of six students were selected from each grade level, including two students with low, medium and high achievement levels. Each achievement level, which depends on grade point averages, consisted of a male and female participant. Then, six mathematics teachers from various socio-economical regions were selected. Both mathematics teachers and preservice mathematics teachers did not attend mathematical thinking lessons.

2.3. Procedures

Mathematical thinking tasks were prepared for each of the four mathematical thinking components described by Mason et al. (2010). The mathematical thinking skills of the students, preservice mathematics teachers and mathematics teachers was examined through their solutions for mathematical thinking tasks by the researcher.

2.3.1. Assessment of students' mathematical thinking

The students solved the four mathematical tasks during the lesson in their own school. An A4 paper was provided for each of the tasks and the students worked on the tasks for about 40-45 minutes. During the implementation period, the classroom teachers and the researcher were present in the classroom.

2.3.2. Teachers and preservice teachers interviews

To conduct in-depth interviews with the participants in a conversational atmosphere, one-to-one interviews were preferred and conducted by the researcher. Four mathematical thinking tasks were discussed during the interviews with the participants. The first stage of the discussions was a section on how teachers and preservice teachers solved mathematical thinking tasks themselves. The participants were first asked to read the task, think aloud about the task, and explain in detail why they thought this way. The second stage was a section on teachers' prediction of middle school students' mathematical thinking. Teachers and preservice teachers were asked to look at the tasks without seeing the student solutions. In the interviews, the tasks were first shown to the participants in order, and each task was discussed separately. At this stage, participants were asked "How do you think middle school students approach and solve this question?". This question aimed to enable teachers and preservice teachers to focus on how students think.

During the interviews, the situations considered important by the researcher were observed and notes were taken. The data obtained through the voice recorders were then converted into text.

2.4. Mathematical Thinking Tasks

Ten non-routine mathematical thinking problems were created and sent to three experts with a Ph.D. in mathematics education, three middle school mathematics teachers, and a language expert. Adjustments were made to the problems in line with the experts' feedback. The main feedback was about the clarity and order of the problems. In addition, enough space should be provided for the students. Necessary adjustments were made in line with these suggestions and each problem was given on an A4 sheet of paper. A pilot study was conducted to evaluate the problems with 60 middle school students. Then, a mathematics teacher and two preservice teachers were interviewed to see the clarity, potential challenges, and duration for the implementation. As a result of these procedures, four out of 10 non-routine problems were decided to be chosen. These tasks were designed to reflect each of the four sub-dimensions of mathematical thinking by Mason et al. (2010), i.e. conjecturing, specializing, justifying and convincing, and generalization.

2.4.1. The task of conjecturing

In the first mathematical thinking task, middle school students, teachers, and preservice teachers worked on conjecturing (Figure 1). This task is based on how many different pencils Ahmet, who wants to buy blue and red pencils from a stationery store, can buy for a certain amount of money. The students were expected to find different possibilities. Moreover, the teachers and preservice teachers were expected to find out the ways the students think.

Figure 1
The task of conjecturing

A stationery shop sells each blue pencil for 2 Turkish liras and each red pencil for 3 Turkish liras. Ahmet bought some pencils from this stationery shop and paid 23 Turkish liras. How many blue and red pencils could Ahmet have bought? Explain how you found the number of items.

2.4.2. The task of specializing

This task is designed to determine the criteria by which students evaluate and use the given values. In general, they are asked to use the given expressions in smaller or special situations. The mathematical thinking task prepared in this context is given in Figure 2. This task asked students to see the relationship between the number of feet and heads in a collection of pictures. Then, depending on this relationship, they needed to determine how many pictures there could be with the total number of heads and feet given in the task. During the study, the task was supported with visuals to make it easier for students to think.

Figure 1
The task of specializing

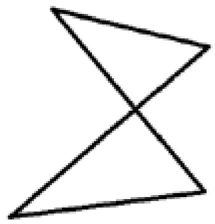
Hilal has a collection of animal pictures. The collection includes pictures of ladybugs, worms, and bees. The number of worms in the whole collection of pictures is more than the number of bees and ladybugs. If there are a total of 10 heads and 18 feet in the pictures, how many ladybugs could Hilal have? Explain your answer in detail. (It is assumed that ladybugs have six feet and bees have four feet.)

2.4.3. The task of justifying and convincing

This task aimed to find out how the students follow a path to achieve the desired outcome based on the given statements (Figure 3). The students are expected to explain their solutions logically.

The task tried to determine how effectively the students used the process of persuasion. This task was inspired from Mason et al. (2010). The task aimed to determine how students find the area of an irregular shape.

Figure 3
The task of justifying and convincing



How can you calculate the area of the figure on the left?
Explain your answer in detail.

2.4.4. The task of generalizing

The task aimed to determine the rules given in the first stage (Figure 4). Based on this rule, it is aimed to determine which situations can occur in the next stages and how they can be expressed mathematically. In the first stage, middle school students were expected to see which rule was given. In the next stages, students were asked to find out what the result would be when applying the determined rule. Determining these rules aimed to encourage students to use mathematical expressions correctly and find more general expressions.

Figure 4
The task of generalizing

Mr. Can has an office on the top floor of a seven-floor business center which has an attendant on each floor. Mr. Can buys a newspaper every day. In this business center, newspapers are distributed according to the following rule: Each attendant distributes half of the newspapers he receives and sends the rest to the next floor up. Since only Mr. Can buys newspapers on the top floor, how many newspapers arrive at this business center each day? Explain your answer in detail.

2.5. Data Analysis

Solutions by middle school students, teachers and preservice teachers were examined respectively. For anonymity, pseudonyms with participant numbers were used for participants. F stands for female, M for male, S for student, PT for preservice teacher, TM for mathematics teacher. A code pool was prepared to determine the mathematical strategies used in the solutions. While defining the codes, a sample solution was coded and the coding reasons were specified. When the generated codes were missing during the coding process, new codes were added. The mathematical thinking tasks were analyzed using a fixed comparison method (Gay & Airasian, 2000) with the obtained code list. The explanations of the codes obtained in this way are given in the table below. Table 1 contains explanations of the terms used during the analysis. In this way, students' solutions were not only evaluated as correct or incorrect, but also the strategy used by the students was analyzed.

The predictions of the teachers and preservice teachers about how the students would solve the same task were examined before moving on to the other mathematical thinking task. They were asked to think out loud about the given task and to express their thoughts on how the students could solve it. In this way, both written and oral data were obtained from teachers and preservice teachers.

The analysis of non-routine problems was made by determining the basic strategies used by the participants. The data obtained in this way was divided into themes and sub-categories by the researcher and the field expert in line with the relevant theoretical information. The researcher and the expert with a Ph.D. in mathematics education worked independently of each other to

Table 1
Strategies used in coding and explanations

Codes	*Explanation*
Estimation and Control	Finding the result of the task by valuing
Establishing an Equation	Reaching the result in the task by establishing an equation
Inequality	Reaching the result in the task by establishing an inequality system
Equation and Inequality	Reaching the result in the task by assessing the equation and inequality together
Using Formulas	Determining which of the existing formulas should be used based on the data
Transforming into Another Shape	Achieving the result by transforming the data into another situation
Utilizing Geometric Features	Using geometric features without formulas or conversions
Working Backwards	Reverse application of the rule by evaluating the data
Logical Reasoning	Associating data with each other by reasoning in solving the problem
Pattern Finding	Creating a rule in a certain order by establishing a relationship between data

determine the themes. The level of agreement between the codes obtained by the researcher and the field expert was determined by Miles and Huberman's (1994) reliability formula. The intercoder reliability coefficient was determined as .94. The data obtained was evaluated together with an unbiased researcher. The discussed cases were examined together and reconsidered.

3. Findings

The findings are presented in four stages: (1) Middle school students' solutions to mathematical thinking tasks, (2) teachers' and preservice teachers' solutions to mathematical thinking tasks, (3) teachers' and preservice teachers' predicting of students' mathematical thinking on the tasks, and (4) comparison of the teachers' and preservice teachers' predictions on the strategies used by the students. The thoughts of teachers and preservice teachers were analyzed from all aspects.

3.1. Middle School Students' Solutions to Mathematical Thinking Tasks

It was first taken into account the students' solutions to the tasks. Each task required a single solution from the students. Consequently, each solution was coded using a single strategy. Table 2 analyzes how the students approach mathematical thinking tasks using different problem solving strategies.

Table 2
Strategies used by students in mathematical thinking tasks

Strategies	*Conjecturing*	*Specializing*	*Justifying and Convincing*	*Generalizing*
Estimation and control	73	62	-	16
Establishing an equation	23	19	-	8
Inequality	-	9	-	-
Equation and inequality	-	6	-	-
Using formulas	-	-	71	-
Utilizing geometric features	-	-	25	-
Transforming into another shape	-	-	15	-
Working backwards	-	-	-	61
Logical reasoning	-	-	-	10
Pattern finding	-	-	-	1

According to students' solutions, they mainly used estimation and control strategies (n = 73) when making conjectures. As can be seen here, they gave values both randomly and systematically in order to complete the task. By experimenting with numbers, they tried to obtain the result without being constrained by any rules. However, systematic valuation can be seen as combining the data based on their relationship when it comes to combining the multiples of the given data, completing or breaking the whole data, and combining the data as a whole. Some students, however, attempted to complete the task by setting up equations (n = 23). By establishing the equations, they attempted to obtain equations using x, y, or different types of variables. By giving values to the variables on the equation, the students tried to solve the equations or find the result by solving the equations they established. Figure 5 and Figure 6 show examples of student solutions to this task.

Figure 5
SM30's solution for the task of making conjecturing-Estimation and control (Mavi: Blue, Kırmızı: Red)

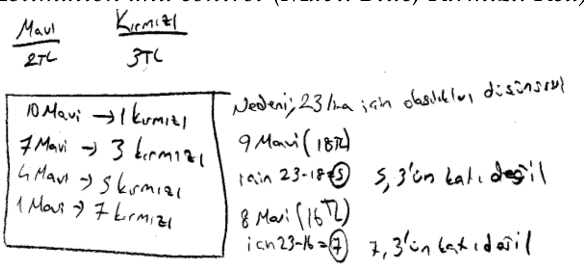
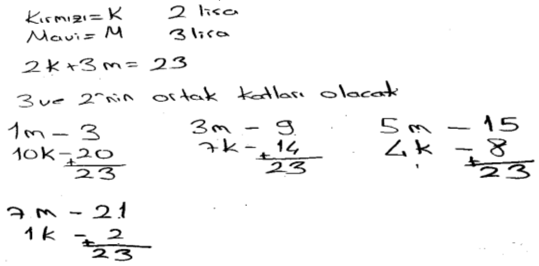


Figure 6
SM31's solution for the task of making conjecturing-Establishing an equation (Mavi: Blue, Kırmızı: Red)



In the specializing task, most students used the estimation and control strategy (n = 62). Their solution was to give random or systematic values to the problem. In this case, a systematic value was given while taking into account the relationship between the data. As they examined the relationship between the data, they broke the whole, completed the whole, and assigned a value based on the relationship between the data. Also, they attempted to tabulate the data, which is different from conjecturing. Aside from that, equations and inequality strategies were utilized. Furthermore, they valued equations (n = 19) by establishing equations with one unknown or multiple unknowns according to the number of unknowns and solving them. Students (n = 9) also attempted to obtain the result by valuing the inequalities they established. Figures 7 and 8 illustrate examples of solutions students have come up with in response to this task.

Figure 7
SM21's solution for the specializing task-Estimation and control

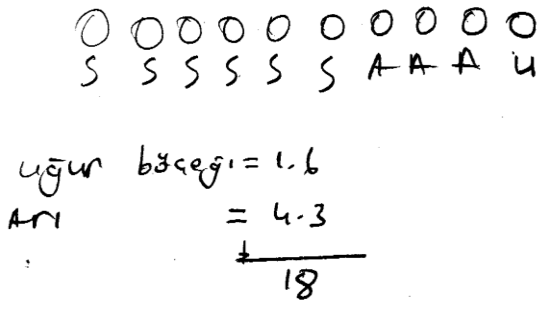
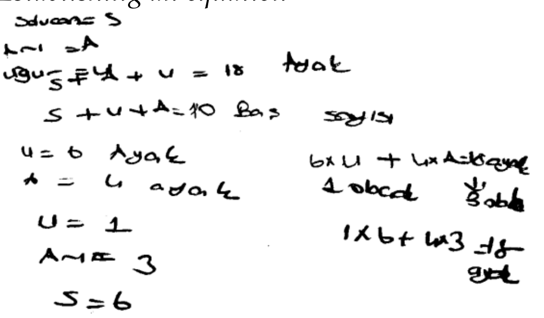
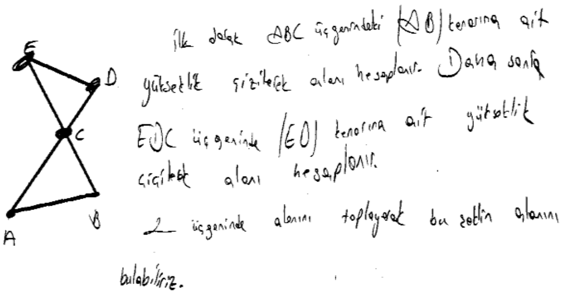


Figure 8
SF19's solution for the specializing task-Establishing an equation



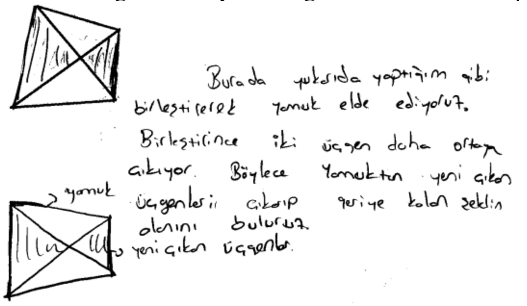
During the justifying and convincing task, most of the students used formulas to come up with solutions (n = 71). Verbally or mathematically, they expressed the formulas they used. They preferred rectangles, trapezoids, and squares here. Students who attempted to use geometric features (n = 25) paid attention to similarity, congruent angles, and angle-side relationships. Using the formulas they obtained, they tried to assign a value to them. The task gave participants the chance to transform the shape provided into another shape (n = 15). As part of this task, students used more than one geometric feature during the task, so solutions were coded using more than one strategy. The following figures show examples of student solutions made within the scope of this task.

Figure 9
SM53's solution for the task of Justifying and Convincing - Using formulas



Translation. First, the height of the side AB in triangle ABC is drawn, and its area is calculated. Then, the height of the ED side of the triangle EDC is drawn, and its area is calculated. By adding the area of the 2 triangles, we find the area of this shape.

Figure 10
SM71's solution for the task of Justifying and Convincing - Transforming into another shape

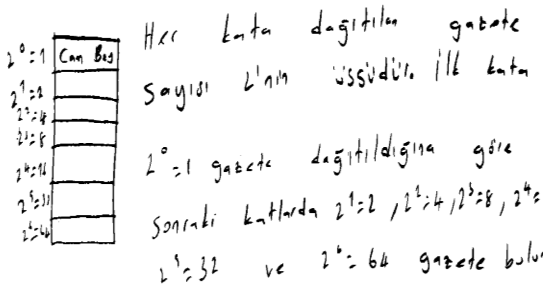


Translation. We get a trapezoid by combining it as I did here. When combined, two more triangles appear. Thus, we subtract the triangles that have just emerged from the trapezoid and find the area of the remaining shape.

Most students attempted to find solutions by working backwards (n = 61) in the generalizing task. In addition, some used logical reasoning (n = 10), pattern-finding (n = 1), establishing an equation (n = 8), estimation and control (n = 16) strategies. In the estimation and control process, the students attempted to give direct values without being systematic. Some participants attempted to solve the equations without providing a value. Figures 11 and 12 show examples of student solutions to this task.

Figure 11
SF34's solution for the Generalizing task - Working backwards

Figure 12
SM87's solution for the Generalizing task - Pattern finding



3.2. Solutions of Teachers and Preservice Teachers to Mathematical Thinking Tasks

Secondly, the process by which teachers and preservice teachers solved mathematical thinking tasks was examined. There was no restriction on the number of solutions they could come up with. Due to this, there were cases where solutions were coded with more than one strategy. Based on the first interviews with teachers and preservice teachers, the findings in Table 3 were derived from the solutions they provided.

It was observed that preservice teachers at all levels preferred to set up equations first when conjecturing. The majority of teachers, however, used equations to establish a relationship between two variables. Equations and inequalities were used by all groups in the specializing task. Using formulas, transforming into another shape, or utilizing geometric properties, all groups reached a solution in the justifying and convincing task. The preservice teachers in grades 3rd and 4th, however, preferred to use pattern finding rather than equations when completing the generalization task. It is important to note that most teachers use the concept of pattern finding here. Working backwards was observed in almost all of the preservice teachers (especially in the first and second years). Further, three teachers used the working backwards strategy.

3.3. Predictions of teachers and preservice teachers on mathematical thinking tasks of middle school students

A mathematical thinking task was presented to teachers and preservice teachers for them to predict how students would solve it. The data from the interviews were analyzed in this way by asking the participants to think aloud during the interview. Table 4 presents teachers' and preservice teachers' predictions about students' mathematical thinking tasks.

As noted in the table above, all of the preservice teachers in 1st grade chose estimation and control or to establish an equation when it came to the conjecturing task. Preservice teachers in 2nd grade stated estimation and control, but one also stated establishing an equation. A majority of preservice teachers in third and fourth grades stated that an equation should be established. Among teachers, estimation and control were mostly preferred. One of the preservice teachers (interviewee #PTM7) commented as: "… A middle school student solves it by trial and error. Tries values starting with the smallest number…". As interviewee #TF1 commented, "…A child who knows the subject of equations well writes the equation down and solves it. Or s/he can solve the problem by valuing them through relating them to price…"

Regarding the specializing task, most of the groups (2nd and 4th-grade preservice teachers, and teachers) stated that students would mostly use estimation and control. One of the teachers (interviewee #TF2) commented as: "…The student will think about the sums starting from either the head or the foot and will reach the result by giving values one by one…"

In the task of justifying and convincing, all groups used the expression of using formulas. Also, it was determined that most of the groups (except 3rd-grade preservice teachers) used the expression of transforming into another shape. One of the preservice teachers (interviewee #PTF4) commented as: "… The students can form two equal triangles. They know the area of the triangle from the formula. Works on two of these and adds them…". As interviewee #TM1 commented, "…Students first try to understand why the given shape is formed. They see that there are triangles here, and they take a step accordingly…"

In the generalization task, on the other hand, they stated working backwards and establishing equations more frequently. One of the preservice teachers (interviewee #PTF4) commented as: "… I think the student goes from a certain number. For example, what would the student say, 'let's say 70 newspapers came'. The student thinks of a number, and thinks of proportion again…". As interviewee #TM3 commented, "…It starts from the last floor. S/he's going to say 'one to the last floor, one to the previous floor. Every time he goes down to the floor, he solves this question backwards…"

3.4. Comparison of the predictions of teachers and preservice teachers for strategies used by middle school students

As shown in Figure 13, teachers and preservice teachers predicted the strategies the students would use, which were compared to the strategies adopted by the students.

Table 3
Strategies used by teachers and preservice teachers in mathematical thinking tasks

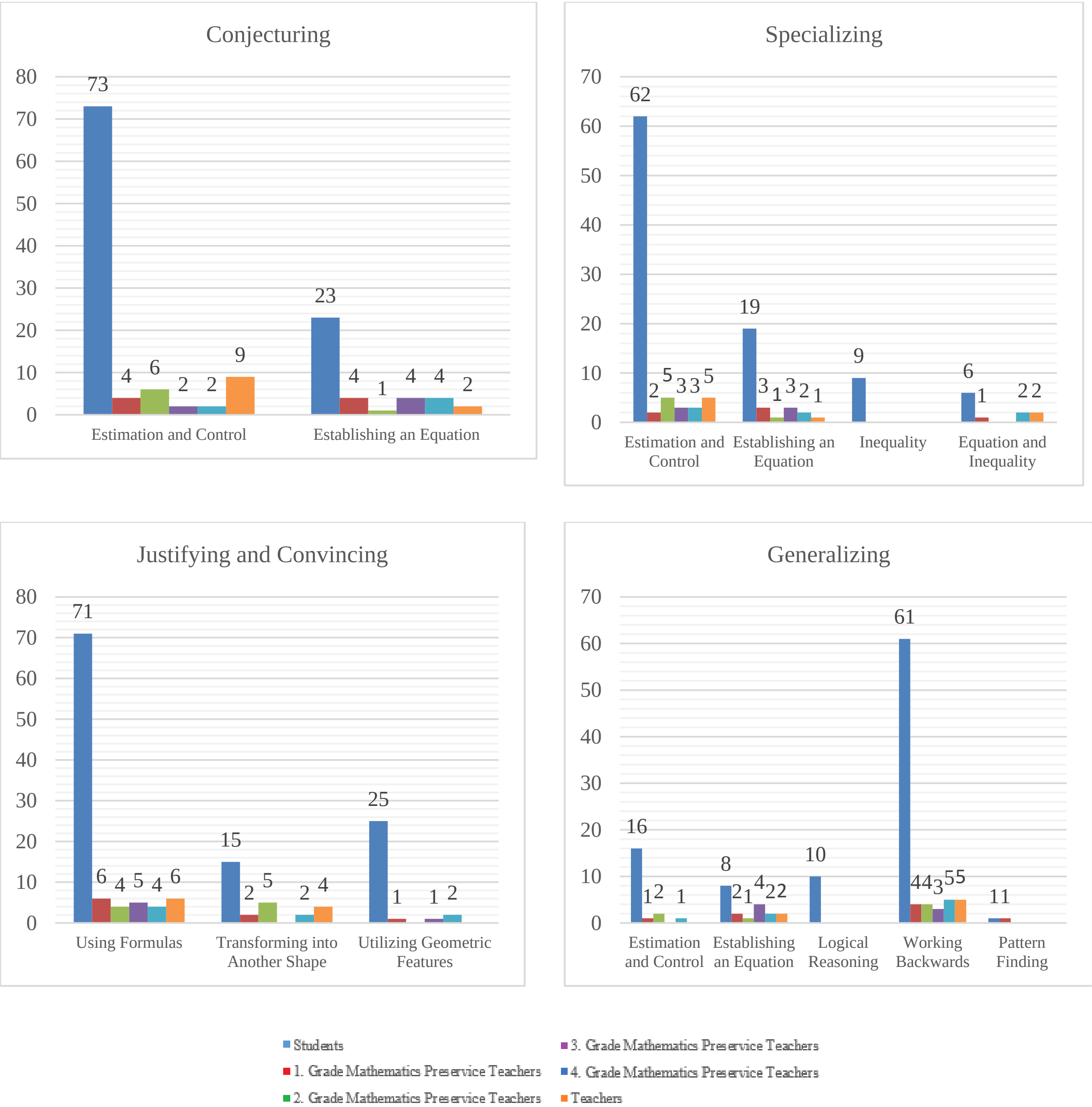
	Levels of Mathematical Thinking Components																			
	Conjecturing					*Specializing*					*Justifying and Convincing*					*Generalizing*				
	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T
Strategies																				
Estimation and control	2	2	3	1	3	-	1	2	1	1	-	-	-	-	-	-	-	-	-	-
Establishing an equation	4	4	4	5	4	1	-	3	-	1	-	-	-	-	-	3	2	5	2	3
Inequality	-	-	-	-	-	2	1	-	1	1	-	-	-	-	-	-	-	-	-	-
Equation and inequality	-	-	-	-	-	3	4	1	4	3	-	-	-	-	-	-	-	-	-	-
Using Formulas	-	-	-	-	-	-	-	-	-	-	4	4	3	4	2	-	-	-	-	-
Transforming into another shape	-	-	-	-	-	-	-	-	-	-	2	4	1	1	3	-		-	-	-
Utilizing geometric features	-	-	-	-	-	-	-	-	-	-	2	-	5	3	4	-	-	-	-	-
Working backwards	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	6	4	2	3	3
Pattern finding	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	1	4

Table 4
Predictions of teachers and preservice teachers on mathematical thinking tasks

	Levels of Mathematical Thinking Components																			
	Conjecturing					*Specializing*					*Justifying and Convincing*					*Generalizing*				
	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T	G1 PT	G2 PT	G3 PT	G4 PT	T
Strategies																				
Estimation and control	4	6	2	2	9	2	5	3	3	5	-	-	-	-	-	1	2	-	1	-
Establishing an equation	4	1	4	4	2	3	1	3	2	1	-	-	-	-	-	2	1	4	2	2
Inequality	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-
Equation and inequality	-	-	-	-	-	1	-	-	2	2	-	-	-	-	-	-	-	-	-	-
Using Formulas	-	-	-	-	-	-	-	-	-	-	6	4	5	4	6	-	-	-	-	-
Transforming into another shape	-	-	-	-	-	-	-	-	-	-	2	5	-	2	4	-	-	-	-	-
Utilizing geometric features	-	-	-	-	-	-	-	-	-	-	1	-	1	2	-	-	-	-	-	-
Working backwards	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	4	4	3	5	5
Pattern finding	-	-	-	-	-	-	-	-	-	-	-	-	-	-	-	1	-	-	-	-

Note. G1: Grade 1; G2: Grade 2; G3: Grade 3; G4: Grade 4; PT: Preservice Teachers; T: Teacher

Figure 13
Comparison of the predictions of teachers and preservice teachers for the strategies used by middle school students



A comparison of the strategies used by the students and the predictions made by the teachers and preservice teachers is presented in Figure 13. As a result, during the conjecturing task, students mostly used estimation and control strategies, and teachers predicted it more accurately than preservice teachers. The specializing task, in which the students mostly using estimation and control strategies, revealed that 2nd-grade preservice teachers and teachers made more accurate predictions than the other participants. Formulas were predominantly used by students in the justifying and convincing task. Teachers and preservice teachers in the 1st grade were found to predict this strategy most accurately. The strategy of working backwards was the most commonly used by students in the generalizing task, while teachers and preservice teachers predicted this strategy the most often.

Based on the results of evaluating teachers' and preservice teachers' predictions together, Table 5 was generated.

Table 5
Comparison of strategies used by students with teachers' and preservice teachers' predictions for mathematical thinking tasks

	Levels of Mathematical Thinking Components							
	Conjecturing		*Specializing*		*Justifying and Convincing*		*Generalizing*	
	T & PTs	S	T & PTs	S	T & PTs	S	T & PTs	S
Strategies								
Estimation and Control	23	73	18	62	-	-	4	16
Establishing an Equation	15	23	10	19	-	-	11	8
Inequality	-	-	-	9	-	-	-	-
Equation and Inequality	-	-	5	6	-	-	-	-
Using Formulas	-	-	-	-	25	71	-	-
Transforming into Another Shape	-	-	-	-	13	15	-	-
Utilizing Geometric Features	-	-	-	-	4	25	-	-
Working Backwards	-	-	-	-	-	-	21	61
Logical Reasoning	-	-	-	-	-	-	-	10
Pattern Finding	-	-	-	-	-	-	1	1

Note. T: Teachers; PTs: Preservice teachers; S: Students

There were some differences between the predictions made by teachers and preservice teachers, as well as the strategies used by students. During the conjecturing task, it was determined that the strategies used by students were similar to those used by teachers and preservice teachers in their predictions. The ratio of students using estimation and control, however, is greater. Several students attempted to use the inequality strategy during the specializing task, but teachers and pre-service teachers did not think students were capable of predicting it. In the justifying and convincing task, the predictions of the teachers and preservice teachers are similar to those used by the students. Students used geometric features more frequently than the predictions of teachers and preservice teachers, however. During the generalization task, students used logical reasoning, but teachers and preservice teachers didn't expect them to do so.

4. Discussion and Conclusion

Students' solutions to the task of conjecturing were first examined within the study. In this task, students mostly used the estimation and control strategies. In various studies, it was observed that students used estimation and control strategies (Altun & Arslan, 2006; Ersoy & Güner, 2014). Using estimation and control, students assigned systematic or random values to the problem. Systematic values were given based on relationships between the data. Specifically, they examined multiples of the values, completed the values into a whole number, or divided the whole to find the result. The reason why they did this is that there are a variety of situations in the given task, especially when estimating and controlling. To complete this task, students had to come up with all the possibilities they could think of. As a result, students tended to use estimation and control strategies more often. Elia et al. (2009) found in their study that students were experienced in using the trial and error strategy. They attributed this situation to the fact that students often use trial and error in everyday situations and mathematical areas. In addition, they stated that they tried to reach the result by making a systematic list of possible solutions to problems and checking the solutions they found. In addition, a small number of students tried to achieve what they wanted by establishing equations. It is thought that the students who tried to use this could see how to set up equations within the scope of the mathematics lesson, and therefore they acted on the idea that it was necessary to set up equations according to the course of the task.

Students used estimation and control strategies mostly in their specializing task. In a similar manner to making conjectures, students provided random or systematic values in order to obtain the result. In particular, they focused on the relationship between the values and tried to break it up or complete it. Alternatively, they tried to create a table using the data provided. Given that trying to reach the result by making a table in problem solving gives students a general perspective (Swafford & Langrall, 2000), it can be said that this situation is important. Here, it is thought that the data within the scope of the task lead students to use such strategies. In addition, it was determined that they also used strategies of establishing equations and inequality. It was observed that having options within the scope of the task led students to use inequality. The students tried to find a solution in this way.

When students were asked to calculate the area of a shape for the justifying and convincing task, they used formulas. Students' preference for formulas can be interpreted as an attempt to memorize the answers. A similar situation has been found in several studies (Greer, 1997; Stacey, 1989; Verschaffel et al., 1994). In addition, the fact that the desired shape was not a regular shape led the students to transform the shape into different forms. It was concluded that they used geometric properties to explain how to express the shapes they obtained. This is similar to the result in Yıldırım's (2015) study that middle school students try to transform the given shapes into familiar shapes such as triangles.

In the generalization task, the students mostly tried to obtain the result by working backwards. It is thought that requiring values after a certain stage within the scope of the task causes students to try to solve the problem in this way. In their study, Altun and Arslan (2006) also found that students used the working backwards strategy in generalization problems. Students used the pattern finding strategy the least. In contrast, Ma (2007) and Rico (1996) found that students tried to find patterns in their studies.

Based on observations of how teachers and preservice teachers approached conjecturing, we concluded that while estimation and control were predominant in newly beginning preservice teachers, establishing equations was primarily expressed by preservice teachers who were more experienced. Estimation and control were most commonly mentioned by teachers. Preservice teachers expressed estimation and control by obtaining the whole and combining the data. As opposed to this, teachers describe situations like combining data, obtaining the whole, and breaking the whole. The fact that teachers are more experienced than preservice teachers may allow them to express different solutions. Accordingly, comparing preservice teachers' professional competencies to student solutions may be important. Teachers explain in detail alongside basic strategies, while preservice teachers demonstrate basic strategies without going into detail of the problem. Studies have also stated that it is normal for expert teachers to have higher analytical skills than novice teachers (Huang & Li, 2012; Miller, 2011). Similarly, teachers' ability to recognize students' mathematical thinking better than preservice teachers is in line with Jacobs et al. (2010). It will be beneficial for preservice teachers to encounter more situations in the process in order to increase their experience. Stockero et al. (2017) also support the idea that preservice teachers' understanding of students' mathematical thinking can improve with practical experience. Lu et al. (2020), on the other hand, emphasize that teachers mentoring preservice teachers is effective for the development of preservice teachers' professional awareness.

The estimation and control strategy was mostly used by teachers and preservice teachers during the specializing task. In this study, it was concluded that 1st-grade preservice teachers used only the expression of assigning a random value to the problems, while in the other groups, teachers and preservice teachers also used the expression of assigning a systematic value to the problems. Both strategies were used by middle school students. Several teachers and preservice teachers reported that students would use setting up equations but value the unknown without solving them. They primarily tried to solve the equations they set up as well as give a value to them. Due to the difficulty of solving equations, teachers and preservice teachers believed that students would value them. Although neither teachers nor preservice teachers expressed using the inequality strategy, students did. In other words, students are able to solve tasks in a variety of ways. Accordingly, El Mouhayar and Jurdak (2013) found that teachers had difficulty explaining students' solution strategies for different variables, although they could recognize students' use of formulas and rules.

Teachers and preservice teachers shared similar views on using formulas and transforming the given shape into another shape as part of the justifying and convincing task. Teachers did not predict that students would be able to use geometric features, despite some preservice teachers predicting that they would. A review of the student solutions reveals a number of formulas, transformations, or geometric features used by the students. The teachers ignored the geometric features even though the students benefited from them. It is believed that preservice teachers have gained practical experience of these problems by passing university entrance exams recently, which makes it easier for them to see different aspects of the solutions. Generally, teachers' knowledge impacts students' ability to think mathematically (König et al., 2014; Schoenfeld, 2011). However, Kaiser and Sriraman’s (2006) study reports that preservice teachers stated that having fresh information enables them to have different perspectives, which is in line with the results of the current study.

Working backwards and establishing equations were the statements made by all groups in the generalization task. Some students used an estimation and control strategy predicted by preservice teachers. This strategy was not predicted by teachers in students' solutions. Furthermore, only a few of the 1st-grade preservice teachers predicted that some students would look for a pattern and obtaining the result. Teachers and preservice teachers would benefit from having more exposure to the tasks within the scope of mathematical thinking's generalization component. El Mouhayar’s (2019) study concluded that the type of generalization problems affect the way teachers notice students’ strategies, which is compatible with this situation. Studies show that it is necessary for preservice teachers to recognize students’ thoughts (Norton et al., 2011), as well as teachers with experience (Nathan & Koedinger, 2000; Xu et al., 2020). In addition, studies have shown that teachers’ having experience does not guarantee that students realize their mathematical thinking (Jacobs et al., 2010; Lee & Choy, 2017).

Teachers’ predicting of students’ mathematical thinking is a component of effective teaching, and, therefore, it is important to support preservice teachers in realizing students’ mathematical thinking (van Es, 2011). This situation brings together the studies on preservice teachers with teachers to a critical point. Although some studies are directed towards teachers or preservice teachers, evaluating preservice teachers’ predictions of students’ mathematical thinking together with teachers will provide more comprehensive results. This situation brings the theoretical and practical lessons of preservice teachers in mathematics education to a critical point (Star et al., 2011). It is thought that the implementation of activities in which teachers can improve themselves in the process can be effective. At this point, this study is thought to provide different perspectives since it evaluates teachers and preservice teachers alltogether.

6. Limitations of the Study and Future Directions for Research

Some limitations should be considered while interpreting the results. The first limitation relates to the participants. A careful selection of participants is vital to the success of any study. Neither the teachers nor preservice teachers took any courses on mathematical thinking, which can be seen as a limitation of this study. Researchers will be able to gain a unique perspective on the field by conducting research with participants who have taken a mathematical thinking course. The written responses were obtained from middle school students, which is another limitation. Students were given the opportunity to think about the problems in this way. While it provides a glimpse into the students' thoughts and decisions, it does not disclose what their thoughts and decisions are during their solutions. Therefore, interviewing the students along with their written answers may be useful for evaluating their solutions in depth.

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