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Research Article

Pre-service mathematics teachers' abstraction of rotational symmetry

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This study investigates first-year undergraduate students of Primary Education Mathematics Teacher Education's knowledge and understanding of rotational symmetry in geometric shapes. Three students participated in this study, which was designed within the framework of a one-to-one teaching experiment (a qualitative research method) in the fall semester of the 2020-2021 academic year. A total of four one-to-one clinical interviews were conducted with the students once a week for an average of one hour and fifteen minutes each. Researchers recorded the sessions with a video camera and kept a log of observations. The data were analyzed and interpreted using continuous analysis and retrospective analysis. Each abstraction type and level of rotational symmetry was assigned indicators based on the results. According to the results, a student at Piaget's experimental abstraction level related to rotational symmetry before the teaching experiment reached the third level of reflective abstraction. Another student at experimental abstraction reached the level of reflective abstraction at level 2; the student who was not at any level reached the level of reflective abstraction at level 1. Students can undertake reflective abstraction by improving their knowledge and understanding of rotational symmetry by using daily life examples and making rotational symmetric drawings in the classroom. Consequently, daily life examples should be used more often in geometry lessons and students should be encouraged to draw more. Additionally, it was suggested that new research be conducted to support student abstraction based on Piaget's reflective abstraction theory.

Keywords: Rotational symmetry; Piaget's abstraction schema; Reflective abstraction; Daily life example

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1. Introduction

The four dimensions of geometry include visualizing and constructing shapes, examining physical phenomena, representing mathematical concepts and relationships, and creating an axiomatic structure as a mathematical system (Usiskin, 1987). It is possible to discover the relationship between geometry and other branches of mathematics using these dimensions. As a result, geometry is defined by the National Council of Mathematics Teachers [NCTM] as a field of mathematics that develops students' reasoning abilities through justification (NCTM, 2000). By using a construction activity, students can examine the definitions and relationships of the concept, question whether the drawing in their mind represents the concept, and thus develop reasoning

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skills. In addition to helping students be able to identify and perceive geometric objects in their environment, geometry also helps them internalize concepts of spatial sense and spatial thinking, which are integral to geometry; by providing context, students can develop reasoning, connection, abstraction, analysis, discovery, critical thinking, prediction, and representation skills (Clark & Otis, 1925, 1927). By viewing geometry from this perspective, one can see how it is frequently encountered in daily life situations as well as how it acts as a bridge between other branches of mathematics (Hartono et al., 2021; Hisar, 2020; Okuyucu, 2022).

Students' content knowledge of geometry is insufficient despite geometry's many applications in daily life, according to French (2004). There is a possibility that this situation arises because geometry involves abstract concepts; another possibility is that abstract concepts are expressed only verbally or theoretically, which may distract students from conceptual understanding. In order for students to develop conceptual understanding, Brenner (2002) argues that they should encounter daily life examples during instruction that they can examine. By creating necessary connections between geometric concepts, van Hiele-Geldof and van Hiele (1984) argue that presenting daily life examples is essential for students to have a conceptual understanding of geometric concepts. Using daily life to teach abstract concepts can be viewed as critical in this context.

The use of daily life examples was found to positively affect students' motivation and enhance their understanding of function in Albayrak et al.'s (2017) study on 3rd grade pre-service mathematcis teachers. A further study by Marchis (2009) provided students with examples of paper snowflakes accompanied by photographs of mosaic patterns belonging to different cultures in order to teach them how to design symmetrical patterns by discovering symmetrical shapes based on line and point, and demonstrated that the examples they examined contributed greatly to their learning. Doruk and Çiltaş (2020) found that using everyday life examples facilitated pre-service mathematics teachers' building connections between mathematical concepts and provided permanent learning in their study aimed at revealing the concept definitions related to sets. These studies raise the question of how student abstractions of rotational symmetry are influenced by daily life examples. Due to the frequent use of symmetry in daily life examples and its connection to many mathematical ideas, this concept was preferred in this study. Due to the importance and significance of establishing connections between rotational symmetry and rotation symmetry for students, it was decided to focus specifically on rotational symmetry. When discussing rotation symmetry, it is important not to ignore rotational symmetry and central symmetry (symmetry with respect to the point), to observe rotational symmetry in geometric shapes, and finally to make abstractions about rotational symmetry to provide conceptual understanding. Based on their existing knowledge of rotational symmetry, the level of abstraction of students was assessed according to how they developed to which level of abstraction after examining daily life examples. By providing an assessment of the abstraction level of rotational symmetry and identifying its indicators, this study should contribute to the literature.

1.1. Mathematical Analysis of the Concept of Rotational Symmetry

Symmetry can be expressed as "a transformation" (Bassarear, 1995) or "a transformation that does not change the properties of the shape when applied" (Leikin et al., 1997, p. 193). Usiskin et al. (2003) defined a symmetrical shape as "F is a symmetrical shape, if there exists a shape F that satisfies the necessary and sufficient condition T(F)=F under a transformation, T". Upon consideration of this definition, it is understood that symmetry has types such as translational symmetry, reflection symmetry, rotation symmetry (Köse, 2012). Based on the components of a geometric object or a mathematical object as input, a movement such as translation, reflection, rotation etc. as transformation action, and the symmetry of the initial geometric object or mathematical object as output, Dreyfus and Eisenberg (1989) stated that symmetry is a bijective function.

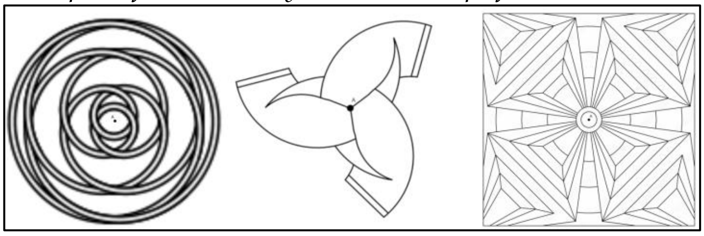
An analysis of the literature on symmetry types shows that there are different classifications about rotational symmetry. Translational symmetry, reflection symmetry, and rotation symmetry are the three types of symmetry classified by Desmond (1997) and Moyer (2001). They conceptualized rotational symmetry as a special case of rotation symmetry. According to Leikin et al. (1997), the four types of symmetry are reflection symmetry, rotation symmetry, translational symmetry, and central symmetry, the latter being a special case of rotational symmetry. Lee and Liu (2012) stated that rotational symmetry is a type of rotation symmetry and listed the symmetry types as reflection symmetry, rotation symmetry, translational reflection symmetry and translational symmetry. Altun (2016), on the other hand, categorised the types of symmetry in the plane into two, as symmetry with respect to the line and rotational symmetry.

Watt (2009) defined rotational symmetry as the situation in which more than one copy of an object with the same position appears when it is rotated by one full turn around a fixed point. In rotational symmetry, the central point of the object is determined as the centre of rotation. Since the main issue is the occurrence of one full rotation, there is no need to identify the direction of rotation. In rotational symmetry, there is a one-to-one mapping between the positions of the points forming the object before and after rotation. In other words, the fact that the object and its image coincide more than once during the rotation action applied to the object for a total of one full turn around the centre point of the object indicates that there is a rotational symmetry and that the object is a rotational symmetric object.

The parameters of rotational symmetry are a fixed point to indicate the centre of rotation, a rotation angle to indicate the amount of rotation and the overlap of the object with itself. Considering the parameters of rotation symmetry are a fixed point to indicate the centre of rotation, a rotation angle to indicate the amount of rotation and the direction of rotation (Zembat, 2013); it can be stated that there is a relationship between the parameters of rotational symmetry. In fact, it can be seen that this relationship explains that rotational symmetry is considered as a special case of rotation symmetry. In rotational symmetry, the images of the object and the object must coincide when rotating around the centre point of the object, this is not always the case in rotation symmetry. In rotation symmetry, when the possible situation(s) of overlapping images with the object during the rotation process occur, essentially rotational symmetry exists. In this study, rotational symmetry is considered as a special case of rotation symmetry. In other words, a rotational symmetry is adopted as a shape characteristic.

It is possible to say that a rotational symmetrical shape has n-fold rotational symmetry if it coincides with itself after being rotated n times around a fixed centre of rotation (Britton & Seymour, 1989). Shapes with n-fold/nth order rotational symmetry will overlap themselves under 360°/n degrees of rotation around a fixed point. Figure 1 includes examples of rotational symmetrical shapes. Since there is a 360° rotation angle for n = 1, the fact that such shapes have 1-fold rotational symmetry reveals that they do not actually have rotational symmetry. Examples are the equilateral triangle and the trapezoid. When the rotation angle is 180°, a special case of rotational symmetry of central symmetry is observed (Altun, 2016). The image obtained as a result of the symmetry of a shape with respect to the origin will be the same as the image obtained as a result of rotating that shape around the origin with a rotation angle of 180°. From this point of view, it is seen that central symmetry and rotational symmetry with a rotation angle of 180° around the origin are equivalent symmetries. Examples of 2-fold rotational symmetric polygons can be given as a rectangle, parallelogram and rhombus; 3-fold rotational symmetric polygon is an equilateral triangle; 4-fold rotational symmetric polygon is a square; 5-fold rotational symmetric polygon is a regular pentagon and 6-fold rotational symmetric polygon is a regular hexagon.

Figure 1
Examples of rotational symmetrical shape from Shutterstock Website



1.2. Theoretical Framework

The Oxford English Dictionary ([OED], 2022) defines the concept as "an idea or principle associated with something abstract". The Turkish Language Association ([TLA], 2022) defines the concept as "the abstract and general design of an object or thought in the mind, an idea, a meaning, concept, or notion". Based on this, it can be said that concepts are abstract mental structures that individuals need to communicate, make meaning of daily life problems and perform thought exercises. Individuals can only be active in the process of bringing sense/meaning by making inferences through concepts. In addition, concepts bring forth the common characteristics and qualities of objects (Dede & Argün, 2004). At the same time, as a symbol of the characteristics specific to certain objects, it enables those objects to be distinguished from other objects.

Specifically, a mathematical concept is "an explanatory model used to explain the observed abilities and limitations of those learning mathematics in terms of their ways of knowing" (Simon, 2017, p. 120). Argün et al. (2014) introduce mathematical concepts among the components of mathematics education. Mathematical concepts are understood as a mental experience through the connection of internal representations in knowledge networks (Godino, 1996). In mathematics education, it is very important for students to make sense of a mathematical concept by making the necessary connections. In this sense, it is necessary to know how students construct mathematical concepts and how their meaning-making processes develop.

Skemp (1986) argues that when individuals encounter an object that they have prior knowledge and notice, they abstract certain invariant properties belonging to these objects, which they perceive as two different objects in a different time, place or situation, and that one of the basic ideas underlying concept formation is abstraction. Davydov (1990, 1972, p. 7) expresses abstraction as "the process of separating a characteristic common to some objects or situations from other characteristics". It can be said that there is a situation of decoupling unnecessary information from among the existing information in abstraction (Hisar, 2020; Tepe, 2022). Starting from here, it can be said that mathematical abstraction is the ability to treat a mathematical concept as a stand-alone object by establishing connections through certain processes and methods that will enable it to stand out from the objects to which it is physically connected. According to Piaget's abstraction schema, the abstraction in which physical knowledge is used is experimental abstraction and the abstraction in which logical-mathematical knowledge is used is reflective abstraction. Piaget (2001, 1977) classified abstraction into two as experimental/empirical abstraction and reflective abstraction and further classified reflective abstraction into three in such a way that there is a hierarchical relationship among them. The levels of reflective abstraction, which is three-fold according to Piaget, are listed as reflecting abstraction, reflected abstraction/reflective thinking and meta-reflection. Zembat (2016) expressed these levels as level 1 reflective abstraction, level 2 reflective abstraction and level 3 reflective abstraction, respectively.

Piaget (2001, 1977) states that a new mathematical concept is never fully acquired through experimental abstraction. In Piaget's experimental abstraction, knowledge is only produced based on the observable properties of objects. The properties of objects are generalised and somewhat summarised (Simon et al., 2004). In reflective abstraction, there is a mental construction process. In fact, Piaget (1980) expressed reflective abstraction as the coordination of actions. Knowledge is produced by focussing on actions performed and the relationships between the actions, independent of the characteristics of the observer and the object being observed. Piaget (2001, 1977) listed the components of level 1 reflective abstraction as i) determining the actions to be performed to make reflective abstraction at a lower level of thought and ii) integrating and reconstructing these actions at a higher level of thought. In level 2 reflective abstraction, a new reflective abstraction process begins by using the knowledge obtained as a result of level 1 reflective abstraction. Reaching the most comprehensive and general knowledge that can be extracted from the knowledge to be abstracted occurs as a result of level 3 reflective abstraction. According to the hierarchical relationship between the levels of reflective abstraction, while the output of one level of reflective abstraction is used as input in the other, level 3 reflective abstraction is the highest level of reflective abstraction. In this study, the structure put forward by Piaget was taken as the abstraction schema.

1.3. Significance and Aim of the Study

Considering that only the teaching of geometry could be attributed to the topic area of symmetry and the basic concepts related to symmetry, which support students' conceptual understanding of shape while developing an aesthetic sense (Olkun & Toluk-Uçar, 2006), should be considered to be limited. As a matter of fact, symmetry encompasses more than just geometry and is closely related to other mathematics topics. Dreyfus and Eisenberg (2000) state that students actively draw upon generalizations about symmetry when they work on algebra, geometry, trigonometry and analysis in primary, secondary and undergraduate mathematics. Therefore, symmetry is the basis for many topics in mathematics at all levels. Several topic areas require symmetry, including equations, fractions, areas, and problem solving (Kaplan & Öztürk, 2014).

Based on a review of national and international literature on rotational symmetry (which can be found in many objects in daily life and is a special case of rotation), no studies directly addressed rotational symmetry; however, rotational symmetry was discussed in many studies. When symmetry should have been gotten with respect to an oblique line, Köse (2012) found that the students gave incorrect answers by getting symmetry with respect to a point. Durmuş (2017) also included an animation about symmetry in his study with eighth grade primary education students, which aimed to address symmetry deficiencies among the animation videos he prepared for them.

From the first grade of primary school, rotational symmetry is a topic included in instruction about symmetry. Rotational symmetry and its relationship with other concepts are therefore important concepts for students to understand. This study, which focuses on the concept of rotational symmetry, seeks to provide a perspective on students' abstraction processes and levels. In addition, the study is expected to contribute to revealing indicators of the concept of rotational symmetry at each level of abstraction. Accordingly, the aim of this study is to investigate the knowledge and understanding of freshman undergraduate students in the Mathematics Teaching Programme about rotational symmetry in geometric shapes selected from daily life examples. In line with this general aim, answers to the following questions were sought:

RQ 1) What are the indicators of the concept of rotational symmetry at each level of abstraction?

RQ 2) What is the development of students' abstraction levels related to the concept of rotational symmetry?

2. Method

2.1. Research Design

In this study, teaching experiment research design was adopted. The philosophy behind teaching experiments is based on radical constructivism. According to the first principle of radical constructivism, knowledge is not passively received from an external source, but is constructed by the subject (von Glasersfeld, 1995). The second principle is that the function of cognition is adaptive and serves the construction of the experiential world (von Glasersfeld, 1995). Teaching experiments have been rapidly adopted as a result of the need for a mathematics-specific model that can be used to explain students' progress in mathematical understanding and instruction and that can eliminate the significant gap between research and instructional practice (Steffe & Thompson, 2000). Clinical interviews are utilised in teaching experiments. Clinical interviews include a series of teaching activities session elements (Cobb & Steffe, 1983), which can be as short as a few hours, a few weeks, or as long as a semester or an academic year (Kelly & Lesh, 2000). In this study, clinical interviews were developed within the scope of teaching activities to be used as a teaching session element. Teaching experiments are conducted one-to-one or in small groups, with a small number of students (Cobb & Steffe, 1983) in order to increase interaction and better reveal student understanding. Cobb (2000) states that teaching experiments in which there is a one-to-one interaction with students are one-to-one teaching experiments. In one-to-one teaching experiments, it is possible to better analyse cognitive structures through the one-to-one interaction with students. Hence, this study adopted a one-to-one teaching experiment design.

2.2. Participants

Study participants were three first-year undergraduates, one male and two females, enrolled in a Primary Mathematics Teacher Education Programme at a state university in the academic year 2020-2021. A male and a female student were also selected for the pilot study from among the students who volunteered. A criterion sampling method, an example of purposive sampling, was used for selecting the participants. Taking no geometry lessons on the undergraduate level was identified as the criterion. The reason behind this was to reveal the amount of knowledge and understanding the students had about rotational symmetry with the education they received until the undergraduate level. Voluntarily participation in the the study was another criterion. The study participants were assigned pseudonames as Ahu, İnan and Yaz.

2.3. Data Collection Tool

Clinical interviews and the researcher's log of observations were used as data collection tools in the study. Teaching activities were carried out in clinical interviews. The teaching activities were developed with an expert mathematics educator and the opinions of another expert mathematics educator were sought. In addition, the data obtained in the four-week pilot study were analysed, necessary revisions were made and the final version of the teaching activities were completed. In the pilot study, it was determined that the students did not use the rulers provided for symmetrical drawings for this purpose, but rather for drawing extension lines when developing the images. Due to this reason, rulers were not provided to the students in the main research. A rotational symmetrical shape consisting only of line segments was given to students in the pilot study, whereas a shape consisting of curves was added in the main study.

The Pre-Clinical Interview Form was used as a pre-test that measured students' knowledge of symmetry. Using Piaget's abstraction schema, this form was developed to determine the level of abstraction at which the students were based on their abilities to observe rotational and symmetry and symmetry with respect to the line in daily life examples as well as to identify symmetry with respect to the line. Decorative artifacts from ancient times, Eiffel Tower, figures of windows and walls were used within the form. Although there are sub-questions on the form, it consists of 11 questions altogether.

As part of the Clinical Interview Form-1, students were asked to provide examples of rotational symmetry in their daily lives and define rotational symmetry with examples of rotational symmetric polygons. Through first knowledge and cases that come to mind upon the mention of rotational symmetry, this form was developed to reveal the students' foundational knowledge of rotational symmetry. In total, there are four questions on the form.

Clinical Interview Form-2 consists of appropriate teaching activities and related questions. The form aims for students to identify the rotational symmetry in the daily life examples provided to them, to specify the parameters of rotational symmetry based on the definition of rotational symmetry provided, to determine which of the given daily life examples are rotational symmetric according to the given definition, to connect the rotational symmetric examples with rotational symmetric polygons, to realise the relationship between n-folded rotational symmetrical shapes. This form was developed to improve students' knowledge and understanding of rotational symmetry and to reveal this development. Daily life examples were selected from Haeckel's Art Forms from Nature (Haeckel, 2004). In total, there are four questions in the form in addition to the sub-questions.

Clinical Interview Form-3 consists of appropriate teaching activities and related questions in which students were asked to complete two different rotational symmetrical shapes, one consisting of line segments and the other consisting of curves by drawing on dotted paper. Additionally there are questions for students to notice the relationship between 2-fold rotational symmetry and symmetry with respect to the line, to design a rotational symmetrical shape on dotted paper, to notice the relationship between rotational symmetry and function, to photograph examples of rotational symmetric in daily life within one month. During the development of this form, the objective was to determine to what extent the students were able to abstract the concept of rotational symmetry through the exercise of drawing within Piaget's abstraction schema at the end of the teaching experiment. There are five questions in total in addition to the sub-questions.

The researcher developed a log of observations to document all activities and ideas that took place during the research process, from collecting the data to analyzing it. During all interviews and immediately after clinical interviews with the students, the researcher kept logs based on observations of the students performing the teaching activities.

2.4. Data Collection Procedure

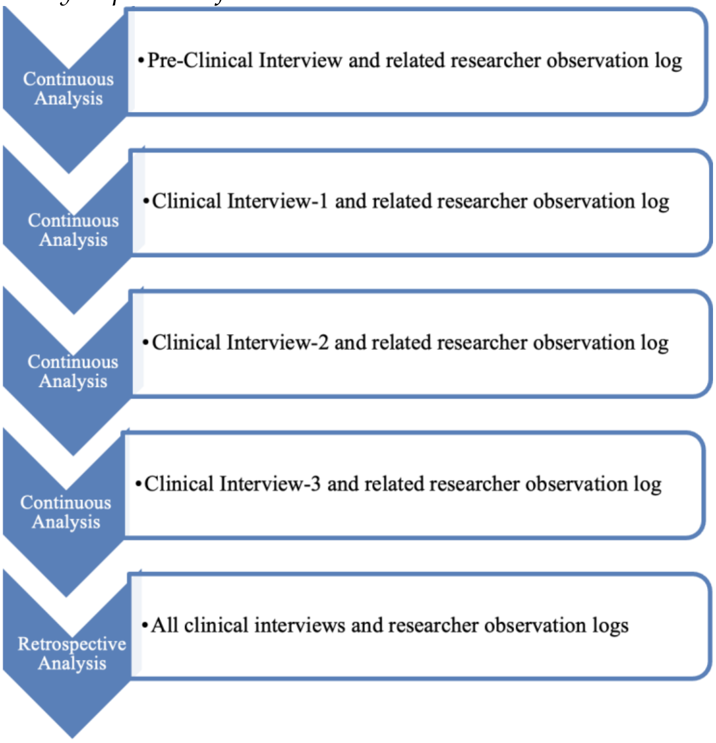
Following the analysis of the data obtained from the pilot study, the data collection tools were revised, following which the main research study was launched. Each student received one teaching session per week during a four-week teaching process. During the first clinical interview, students were asked about their prior knowledge of symmetry. In all other clinical interviews, teaching activities designed as one-to-one teaching experiments were carried out. Pre-Clinical Interview Form, comprising questions in the form of a pre-test, was administered to students in the first week. A clinical interview was conducted in the second week in which questions related to the teaching activities in Clinical Interview Form-1 were asked. A clinical interview was conducted in the third week during which the clinical interview form-2 was used to ask questions about teaching activities; and a clinical interview was conducted in the fourth week, which was the last session, when the clinical interview form-3 was used to ask questions about teaching activities. Video cameras were used to record all clinical interviews, allowing researchers to re-examine the data by referring back to the recordings to verify that no points were overlooked. An overall total of 225 minutes were spent interviewing Ahu, 211 minutes with İnan, and 197 minutes with Yaz.

2.5. Data analysis

In order to analyze the data obtained, thematic analysis was used. The continuous analysis and retrospective analysis were used in this context, as shown in Figure 2. An educator with expertise in mathematics was present during the analysis of the data. In addition to the researcher's observations recorded after each interview, all four clinical interviews with the students were subjected to continuous analysis. Retrospective analysis was used to analyse the data obtained from the students' explanations to the questions in all clinical interview forms. The audio of the video recordings was first transcribed into the Word environment without any changes during the data analysis. Furthermore, screenshots of the students' drawings were taken, and the data was transcribed without altering the order of monologue/dialogue/visuals. Coding was then performed on the transcribed data. Piaget's abstraction schema was used to place the student behavior that emerged during the formation of rotational symmetry at the appropriate abstraction levels. For abstraction levels, behaviour-indicators were assigned and the students' levels were determined.

Figure 2

Analysis process of the data



2.6. Validity and Reliability of the Study

Guba and Lincoln (1982) classify validity and reliability for qualitative research into four components: credibility (internal validity), transferability (external validity), reliability, and confirmability (objectivity).

In order to ensure the credibility of the study, interactions of long duration were established with the students through clinical interviews. Additionally, each student was observed continuously by the researcher in a log, a field expert was consulted to determine if the interpretation was accurate, peer feedback was provided to include interpretations beyond the researcher, and instead of using only one source of data, each clinical interview was video recorded by triangulation.

Upon determining the appropriate level of education for the study, purposive sampling was conducted through voluntary participation to ensure its transferability. Based on video recordings of clinical interviews and the researcher's daily observations, the video recordings of the interviews and the environment were used to describe the students and the environment in detail.

In order to ensure reliability, the study's method was explained in detail and data was analyzed by a field expert in addition to the researcher.

Finally, for confirmation of the findings, triangulation was established as mentioned above; clearly defined roles were assigned to the researcher and the other researcher as a field expert, researcher bias was reduced, methods of obtaining results were explained in detail, and student papers and verbal statements were included in the research results section, which allowed for a control of confirmability.

3. Results

3.1. Results Related to Experimental Abstraction

In the Pre-Clinical Interview, the students were given daily life examples that were rotational symmetrical or contained rotational symmetrical motifs with aim of examining the symmetricity. Ahu and İnan were able to intuitively notice rotational symmetry in the examples. Due to the fact that they were unaware that this geometric concept was rotational symmetry, they were not able to express it in this way. Observations showed that both described rotational symmetry in informal terms. Based on the observable features of the shapes in the examples, they made a summary based on their experiences. It was of interest to see that in the examples that had rotational symmetrical motifs their foremost attempt was to seek out symmetry relative to the line. According to Zembat (2016), Ahu and İnan are at the level of experimental abstraction. Experimental abstraction uses only physical knowledge about rotational symmetry. The student Yaz was unable to notice rotational symmetry in daily life examples, even intuitively. Consequently, Yaz did not display any behavior required by the experimental abstraction level, which produces knowledge based on observable properties of the shapes in examples. As a result, Yaz was not at a level of abstraction that was experimental or reflective when it came to rotational symmetry.

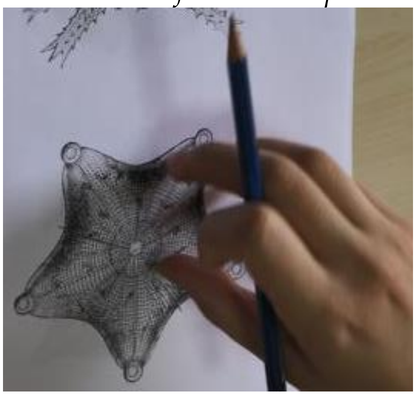
3.2. Results Related to Reflective Abstraction

3.2.1. Results related to level 1 reflective abstraction

In Clinical Interview-1, the student Yaz, responded to the question of what comes to her mind when she thinks of rotational symmetry as: “Actually, nothing. I have never heard of rotational or the concept.” When she was asked about an example of a rotationally symmetrical object from daily life, she gave a similar answer, and Yaz was not asked about the definition of rotational symmetry. After being asked to analyze certain polygons based on rotational symmetry, Yaz responded, "My brain just stopped. I’ve never heard of rotational symmetry and have no idea why I haven’t heard of it.” and could not analyze the polygons.

In Clinical Interview-2, they were asked to analyze daily life examples for rotational symmetry. Initially, Yaz noticed symmetry only with respect to the line and was unable to explain rotational symmetry. Her analysis of daily life examples led her to make inferences about rotational symmetry. According to her, the daily life example similar to a star caught her attention as follows: "Parts like this arm are things that are constantly repeated within themselves. If I take this piece entirely, I can say that it is repeated. Taking them as a complete piece, they all repeat the same way." The student illustrated this by demonstrating it by rotating her hand in Figure 3. A number of inferences were made by the student regarding rotational symmetry. According to the student, rotational symmetry consists of equal parts: "I could say it is a shape with equal parts. Perhaps that's not quite right, but when I talk about rotational symmetry, it feels like repetition and placing the equal parts again." Yaz stated that a shape has equal parts and that those equal parts repeat themselves. Based on this explanation of Yaz, it was evident that the concept of rotational symmetry was being abstracted.

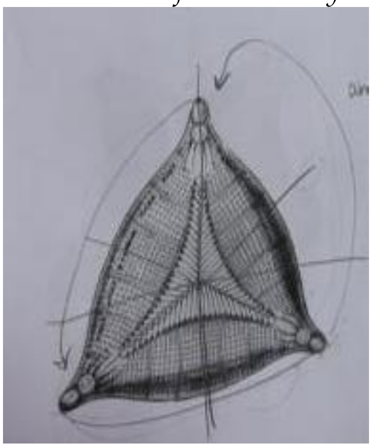
Figure 3
Examination of Yaz on the possibility of the daily life example having rotational symmetry



She was then asked to analyse the definition of rotational symmetry. She expressed that her initial reaction was similar to her own: "As I said here, repetition is actually true." She replied to the question of identifying rotational symmetry parameters based on the definition by saying, "Those repetitions are the rotation of a point and a shape by an angle. I could only say this." It was observed that she could not completely determine the parameters of rotational symmetry, and only the repeating parts and the rotation relationship between these parts caught her attention. As a result, Yaz recalled the rotational symmetry of the arms in the previous clinical interview and said, "If it is like an arm, then it is rotational symmetry."

On the basis of rotational symmetry, Yaz was asked to reexamine the symmetry of daily life examples. Unlike her previous examination, this re-examination was noted to identify a rotational center and congruent parts, as well as the rotational transformation between them. She divided the fourth daily life example into three equal parts while analysing it and said, "When I break it down into parts, it repeats. It is rotated according to an angle. There is a rotation on all of these pieces. I don't know how many angles there are, but I can say it is rotational symmetrical." She indicated the rotation relationship by drawing arrows as seen in Figure 4.

Figure 4
Examination of rotational symmetry of the fourth daily life example by Yaz



Yaz, who could not express an opinion about rotational symmetry before seeing the definition of rotational symmetry and examining the daily life examples, was asked to examine the polygons. Yaz focused on finding a rotational relationship in polygons by determining their centers of rotation, dividing them equally, and finding the rotation between equal parts. Using a triangle as an example, she divided it into equal parts and explained it as follows: "I think the triangle guarantees this slightly. Having different sides made it impossible for me to divide it into equal parts. It is not rotational symmetric." Her examination of the square led her to determine its center, draw the diagonals, and divide it into four equal parts and declare, "The square can be divided, yes. These are all equal. That's why this triangle has always been rotated, first here, then here, then here. Therefore, it is rotational." Yaz did not explain angles in this process. Her next question was to determine whether she could connect angles to shapes by asking "You said that the equilateral triangle is rotational symmetrical. What can you say about the angles of rotation?". She answered as "120 degrees for each, as one full turn". In order to determine if she could connect overlaps with polygons, she answered the question "How many times does it overlap with itself when it makes a full turn? " as "I divided it into three parts. A three-fold overlap will occur. Actually, I think it is also important how many pieces we divide it into. I would have overlapped them twice if I divided them into two pieces." She illustrated the angles and rotation relationship in Figure 5 with arrows. It was possible for Yaz to find the rotation angles of some rotationally symmetric polygons in a complete manner, but not for others such as regular pentagons. Consequently, Yaz was unable to make the connection between the congruent part and overlap number and could not abstract.

Figure 5
Examination of overlap within the equilateral triangle by Yaz

She was asked if she could make any connection between rotational symmetry and function. Using a coordinate plane, Yaz drew two triangles in different regions and said, "If this is reflected according to the x-axis, for example, a shape like this will be formed. Take the point (-2,2) as an example. (-2,2) will also apply here. I mean, only the y will be changed here." After saying so, she added, "How can a function be fitted to each shape? I am not sure about that. That's why I don't think all of them can be called functions. Because there are many more different shapes." It was observed that Yaz could not discover the relationship between rotational symmetry and function, and therefore could not abstract it.

In order to assimilate the abstractions related to rotational symmetry, the student was asked to design a rotational symmetrical shape on dotted paper. Yaz was able to design a 4-fold rotational symmetrical shape as shown in Figure 6. In addition, Yaz was asked to take a photograph of the examples of rotational symmetrical objects from daily life within the one-month period. She took various photographs such as flowers from nature, lace, carpets and mats, chandeliers, lampshades, and bed linen sets from her close environment, and wooden boxes from decorations. At the end of the teaching experiment, it was determined that Yaz could not fully determine and comprehend the parameters of rotational symmetry and could not make connections related to rotational symmetry. In other words, although she was able to carry out a retrospective thematization process as a construction process (von Glasersfeld, 1991) at the thought level, it was found that Yaz was able to only reach level 1 of reflective abstraction about rotational symmetry as a result of her inability to obtain the most comprehensive and general abstraction knowledge necessary for the formation of the concept of rotational symmetry.

Figure 6
Rotational symmetrical shape designed by Yaz

3.2.2. Results related to level 2 reflective abstraction

In Clinical Interview-1, İnan expressed when rotational symmetry is mentioned, "a symmetrical object" that comes to one's mind. Then, he drew a variety of triangles on a dotted paper and said, "We can call the rotation of any triangle around a certain point as a rotational symmetry." It can be said that İnan made an informal definition of rotational symmetry. He gave the example of a helicopter propeller and wheel as examples of rotational symmetrical shapes from daily life.

When asked to define rotational symmetry, İnan stated as follows: "We can call rotational symmetry the rotation around a certain symmetric object by maintaining the distance between the point we take on or around it and the object." When this definition is analysed, it can be seen that İnan actually did define rotational symmetry and expressed some of the parameters of rotational symmetry.

It was observed that İnan, who was asked to analyse the rotational symmetry of certain polygons provided before the definition of rotational symmetry was given, focused on whether the polygon was a regular polygon or not. For example, after examining the equilateral triangle, he said that he thought that it was not rotational symmetric: "If it were an equilateral triangle, we would see that no matter how much we rotate it on a certain centre, its angles would not change. But in a scalene triangle, the positions of the angles change." He stated that he thought that a regular pentagon was rotational symmetrical by pointing to its centre point and said, "Assuming that it has a centre because it is regular, we see that its shape does not change when we rotate it clockwise or counter-clockwise on both the centre of gravity and the centre of mass". Here, it can be said that he informally stated that orientation would not be changed as a result of rotation movement. When asked what he could say about the angle of rotation in relation to a certain rotation movement, İnan said, "The angle does not make any difference. Whether we rotate 60° or 90° or 360°, because it is regular". From this point, it was seen that he had conceptual deficiencies regarding rotational symmetry.

In Clinical Interview-2, the student was asked to analyse the rotational symmetry of certain daily life examples. At first, it was observed that İnan connected the daily life examples with polygons and thought whether they were rotational symmetric or not. When it came to the daily life example seen in Figure 7, he stated as following:

> "When we take this as the centre, I think that this and this are two different motifs. When we turn it, it is not rotational symmetrical because we see it differently from our point of view. If these were the same, I would say rotational symmetrical. Actually, I can think of it as a regular polygon, after all, it is a circle. But it can be rotational symmetric even though it is not a regular polygon."

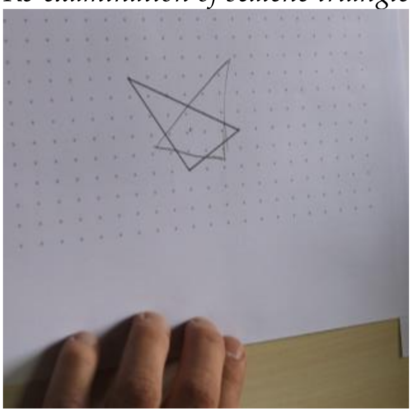
In addition to stating that he thought it was not rotational symmetric, it was also observed that he stated the condition that must be met for it to be rotational symmetric. Here, it was noteworthy that for the first time, İnan stopped focusing on being a regular polygon in the daily life example he analysed and focused on congruent parts. The reason why he thought that it was not rotational symmetric was that he did not consider the angles of rotation. It was determined that İnan, who completed the examination of daily life examples, started to make abstractions about rotational symmetry, albeit at a low level. A change was observed in İnan's ideas and methods while analysing the first and last examples. In other words, it was determined that situations that attracted his attention about rotational symmetry occurred according to the differentiated daily life examples.

Figure 7
Examination of the seventh daily life example by İnan


İnan was asked to read and analyse the definition of rotational symmetry and state the parameters of rotational symmetry. His comment was that, "It says that the object should be rotated around a fixed point. This is an important point. It says that it should be rotated by a certain angle. In other words, it should rotate 360° and reach the same position." It was seen that İnan was able to determine that there should be a centre of rotation and that the direction of rotation was not important. However, he could not make a clear inference about the angle of rotation and could not determine that the image of the shape should coincide with itself.

İnan was asked to analyse the rotational symmetricity of daily life examples in line with the definition provided. Unlike his first analyses, İnan focused on finding a certain rotation angle around a fixed point. For example, in the fifth daily life example resembling a starfish, he stated the rotation angles correctly and explained as follows: "I think it is rotational symmetrical since its appearance will be the same when we rotate it by 72° and its multiples." From this point, it was determined that he started to make abstraction at a higher level as he examined daily life examples, as he mentioned not only 72° but also its exact multiples.

Then, he was asked to re-examine the rotational symmetry of certain polygons given in line with the definition of rotational symmetry. As in daily life examples, İnan focused on finding a certain rotation angle around a fixed point in polygons. For example, by making the drawing shown in Figure 8, he stated that it was not rotational symmetric. He had a different perspective this time in explaining the overlap situation: "When we rotate this triangle by 90°, I see that it is like this and it does not match one-to-one. Again, I think it is not rotational symmetric."

Figure 8
Re-examination of scalene triangle by İnan


While examining the rotational symmetry of polygons using the definition of rotational symmetry, İnan generally tried to find a certain angle of rotation around a fixed point and acted based on the internal angles of the polygon. However, although he determined the angles that provide rotational symmetry in some polygons (rectangle, parallelogram and rhombus), he stated that he thought that they were not rotational symmetric, suggesting that they did not provide rotational symmetry at 90°. When asked why he thought they should be at 90°, İnan could not provide a logical answer. Based on his explanations and drawings, it can be said that the fact that he stated that it should be provided at every angle in daily life examples and that it should be provided at 90° in polygons shows that the abstraction process related to rotational symmetry is limited in İnan. This may be due to the prototype effect caused by the use of the same shapes in similar positions as examples in symmetry teaching. On the other hand, in response to the question of how many times the images of rotational symmetric polygons would overlap with themselves, the students answered "Three overlaps occurred and it was named triangle. Here, six overlaps occurred and it was called a regular hexagon. This is a square and it is actually a regular quadrilateral, but its special name is square." It was observed that he was able to abstract the relationship between n-fold rotational symmetrical shapes.

In Clinical Interview-3, a rotational symmetrical shape consisting of curves but not rotational symmetric was asked to be transformed into a rotational symmetrical shape. İnan was able to obtain a 2-fold rotational symmetrical shape by utilising symmetry with respect to the line. The dialog is as follows:

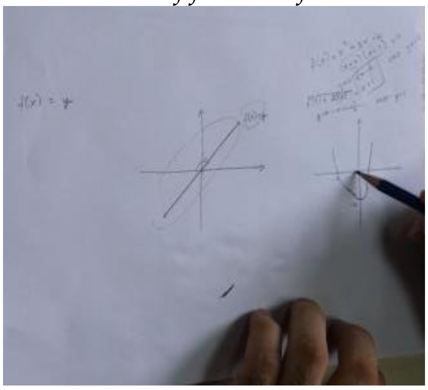
> Researcher: I saw that while making a 2-fold rotational symmetrical shape, you determined an axis and drew it by taking perpendicular distances according to it. Is there a case that attracts your attention here?
>
> İnan: In 2-fold rotational symmetric, we can say that the mutual sections are symmetrical to each other. There is nothing I can say other than that."

From this comment, it was observed that İnan could not establish the necessary connection.

> Researcher: Can you make any connection between rotational symmetry and function? Do you think rotational symmetry is a function? If it is a function, why is it a function? If not, why is it not a function?
>
> İnan: [Asking for a paper to draw on, he drew the line y=x] Let's draw the line $f(x) = y$ like this. If we take the centre, I see that this point gives the same image when it rotates 180°. I think that the graph is suitable for rotational symmetry. That's how I can connect.
>
> Researcher: Can you generalise this for all functions?
>
> İnan: Let's take the function $f(x) = x^2 + 3x - 4$ as an example. [He found the roots and drew its graphical image] I see that it does not give the same image when it rotates at a degree other than 360°. So I think that if only a symmetric graph is formed for the function, it would be suitable for rotational symmetry.

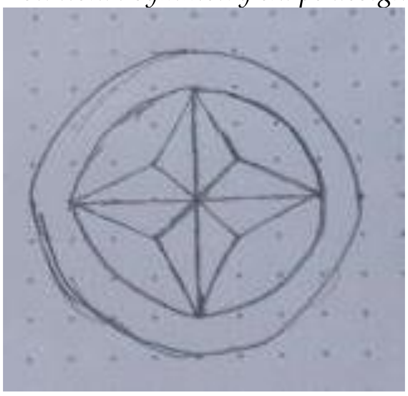
When the drawings and explanations were analysed, it was seen that İnan rotated the graph of the function $y = x$ by 180° around the origin and stated that it overlapped with itself and that it was related to rotational symmetry. In other words, he tried to explain the function he gave as an example from the graphical image. In the function $f(x) = x^2 + 3x - 4$, it was seen that he stated that it overlapped with itself by rotating only 360° around the origin and for this reason, it was not related to rotational symmetry, in other words, he tried to explain it from the graphic image of the function he gave as an example. Based on all these, it was seen that İnan could not abstract the relationship between rotational symmetry and function at a sufficient level. In other words, İnan could not connect rotational symmetry with the rotational transformation, which is a bijective function.

Figure 9
Examination of function for rotational symmetry by İnan



When asked to design a rotational symmetrical shape on the dotted paper given to him, İnan designed a 4-fold rotational symmetrical drawing as in Figure 10. The fact that İnan designed a shape based on a car wheel and a rim shows that he thought of daily life examples. In addition, İnan was asked to take photographs of the objects that attracted his attention as a rotational symmetrical by examining daily life examples in his surroundings and to send them within one month. He took photographs of different types and colours of flowers from nature, a car wheel, lace, a lemon squeezer, a lamp and saltshaker. As a result of the teaching experiment, it was determined that İnan was able to determine and comprehend the parameters of rotational symmetry by using the logical-mathematical knowledge type (Zembat, 2016), but could not make all connections related to rotational symmetry. In other words, although he was able to distinguish thinking as a retrospective thematization process from construction process (von Glasersfeld, 1991, p. 12), it was determined that İnan was able to progress to level 2 of reflective abstraction about rotational symmetry as a result of his inability to obtain the most comprehensive and general knowledge that can be extracted with the knowledge to be abstracted, which is necessary for the formation of the concept of rotational symmetry.

Figure 10
Rotational symmetry shape designed by İnan



3.2.3. Results related to level 3 reflective abstraction

In Clinical Interview 1, Ahu described the situation that came to her mind when she thought about rotational symmetry: “I thought of it as the 90° symmetry of a square with respect to a line. It's like doing a rotation. Or it is like getting its symmetry according to any angle.” She defined rotational symmetry informally by emphasizing the angles. When asked to elaborate on the alpha angle she mentioned, Ahu said that she remembered the rotational symmetrical shape consisting of the arms she had examined in the Pre-Clinical Interview: “In our previous interview, there were three arms, somehow. At first, I was thinking like 90°, 180°, 270°, 360°, but with those arms, I thought it could be any alpha angle. Even 1°, even 2°.” From this point, it was seen that Ahu started to abstract with the connections she made about rotational symmetry. She presented examples of rotational symmetrical shapes from daily life such as flowers, a pinwheel, and windmills.

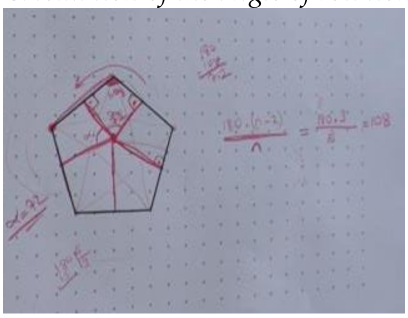
When asked to define rotational symmetry, Ahu explained her thoughts as follows:

> "For example, when the hour hand moves at an angle of this many degrees in a clock, it falls here. It designs rotational symmetry by moving little by little. I mean, I thought it was like the shapes that we can move up at an angle of alpha".

Considering her drawings and explanations, it was determined that Ahu did not have any logical-mathematical knowledge about rotational symmetry at the beginning of the interview, and she tried to explain rotational symmetry based on symmetry with respect to the line and rotational symmetry based only on her guesses. Afterwards, as questions about rotational symmetry were asked, it was determined that she connected it with the shapes in the pre-interview and started to make abstractions and made explanations about rotational symmetry.

While examining whether the given polygons were rotational symmetric or not, Ahu focused on the congruent parts formed by decomposing the polygons. For example, she stated that she thought a regular pentagon was rotational symmetric as, "If I determine a center, I can show a continuous rotation with equal angles from this center. Therefore, yes." From this comment, it was found that she started to abstract the concept of center of rotation in rotational symmetry. When asked what she could say about the angles of rotation, as can be seen in Figure 11, she stated only 72° with the calculations she made based on the interior angles, and did not specify an angle related to integer multiples.

Figure 11
Calculation of the angle of rotation in a regular pentagon by Ahu



It was observed that Ahu focused on determining a rotation angle in line with her previous inferences while examining the rotational symmetry of the daily life examples given in Clinical Interview-2. For example, in the second daily life example, which had a spiral structure and was not rotational symmetrical, she said, "When we take the whole shape and rotate it 360°, we get the same shape. So then all shapes would be rotational symmetrical. That's not the reason why." and stated that she thought it was not rotational symmetric. When Ahu's explanation was analysed, it was determined that she could not determine a rotation angle smaller than 360°, and in addition, she unknowingly made a non-rotational symmetrical shape matching, which is also expressed as a 1-fold rotational symmetrical shape. It was determined that Ahu, who continued to make abstractions as she continued to examine the daily life examples, made correct determinations about rotational symmetry, especially in the last daily life examples, and also started to specify all rotation angles completely.

When Ahu was asked for her opinion about rotational symmetry as a result of the definition of rotational symmetry provided, she stated, "At first, I always thought that the angles were fixed, like 90°, 180°. Then, as I saw examples from daily life, I realized that it could be any angle. My thoughts became clearer and overlapped with the definition provided here." Regarding what she could specify as its parameters, she answered as "When we rotate a part of a shape to make a complete 360° turn, those parts overlap again." Based on the examples Ahu examined as well as the definition given, it was determined that her ideas about rotational symmetry became clearer, but she had inadequacies such as not being able to determine the center of rotation, not being able to determine the angles of rotation, and not being able to express it as an n-fold rotational symmetrical shape due to her inability to make abstraction at a sufficient level.

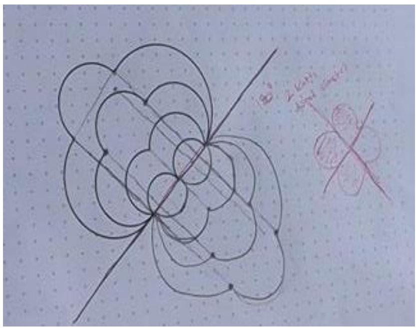
Ahu was asked to re-examine daily life examples based on her abstractions of rotational symmetry and the definition of rotational symmetry. Ahu focused on determining an angle of rotation similar to her previous investigation and gave exactly the same answers as her previous answers. Then, she was asked to re-examine polygons. She stated, "I think it will not change, because before analyzing daily life examples, my thought for the angles was 90°, 180°, or something clear. But I think it can be any angle." Ahu wanted to re-state that she made abstraction regarding the angles of rotation. In fact, Ahu gave the same and correct answers regarding whether the polygons were rotational symmetric or not. On the other hand, she mentioned any rotation angles in all rotational symmetric polygons. In addition, it was observed that Ahu abstracted the case of n-fold rotational symmetrical shapes by stating that integer multiples of the smallest rotation angle would also be valid in rotational symmetrical shapes and connecting this with the number of overlaps.

In Clinical Interview-3, Ahu was able to realize this process by making use of symmetry with respect to the line in the shape consisting of curves, which is not symmetrical in its current state but can be made rotational symmetrical by drawing. The dialog for this process is as follows:

> Researcher: You realized your drawing by making use of symmetry with respect to the line. At the same time, you said that this shape is a 2-fold rotational symmetrical shape. Do you think there is a relationship between these two?
>
> Ahu: This is the reflection symmetry of this (see Figure 12). In rotational symmetry, it is as if we always get the reflection symmetry of the shapes. But it is as if there is always a reflection symmetry between the parts of the shape.
>
> Researcher: Is it between all parts of all rotational symmetrical shapes? Or can it be customized?
>
> Ahu: [Considering a 4-fold rotational symmetrical shape she drew on the side] For example, when I take two of them together, it is still reflection symmetry. But not when I take three. I mean, only when there is a 2-fold rotational symmetrical shape, this situation exists.

From these drawings and explanations of Ahu, it was determined that she was able to abstract the relationship between symmetry with respect to the line and 2-fold rotational symmetry.

Figure 12
Examination of 2-fold rotational symmetry shape regarding the relationship with rotational symmetry by Ahu

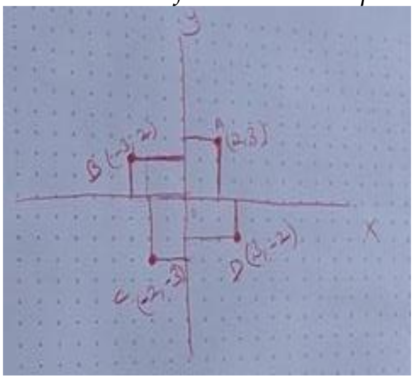


In regard to whether she could connect rotational symmetry with a function, she thought for a while and said, "When I take a point and apply rotational symmetry with a 90°, its places change. I think that it may be a function by thinking that it is related to x and y in the coordinate system with the points." Following this comment, the student made the drawing seen in Figure 13 and stated the following:

> "We can show it as $f(2) = 3$. When we design complete rotational symmetry with a 90° rotation, point A of the shape overlaps with point B. Actually, they are all point A, but I did it like this to make it understandable. Their places have changed and they will go back and forth between two and three. Here $f(-3) = 2$. Here $f(-2) = -3$. Here $f(3) = -2$."

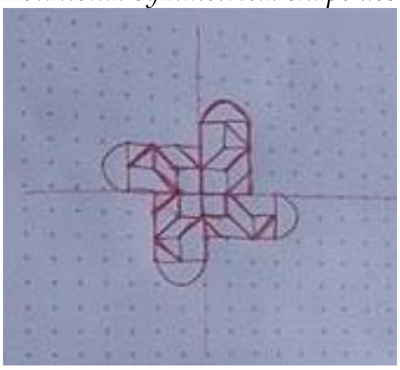
It was observed that Ahu marked the coordinates of this point A as points B, C and D, respectively, on all equal parts of the 4-fold rotational symmetrical shape and indicated the coordinates of these four points with function notation. In response to the question "So, is rotational symmetry and function related?", the student answered, "It is related. When I rotate 180°, I get minus x, minus y, x, y, minus x, minus y. When I rotate 270°, it becomes x y, y, minus x." In this context, it was determined that Ahu abstracted the relationship between the function and rotational symmetry by connecting it with rotational transformation, which is a bijective function.

Figure 13
Examination of the relationship between function and rotational symmetry by Ahu



When asked to design a rotational symmetrical shape on dotted paper, Ahu stated that she was excited to make such a drawing and that she would base it on a pinwheel by using her imagination. Then, she designed her drawing as 4-fold as shown in Figure 14. Within a one-month period, she photographed and sent many different kinds of objects such as flowers of different types and colors from nature; car wheels, lace, carpets and mats, fans from her close environment; and bathroom tiles, wall motifs, iron doors, window railings from decorations. At the end of the teaching experiment, it was revealed that Ahu was able to determine and comprehend the parameters of rotational symmetry and realize all connections related to rotational symmetry. In other words, by using her logical-mathematical knowledge (Zembat, 2016), she was able to perceive this concept as a result of the most comprehensive and general knowledge that can be extracted with the knowledge to be abstracted, which is necessary for the formation of the concept of rotational symmetry through reflective abstraction (Piaget, 2001, 1977, p. 6). Based on this, it was determined that Ahu was able to progress to level 3 of reflective abstraction about rotational symmetry at the end of the teaching experiment process.

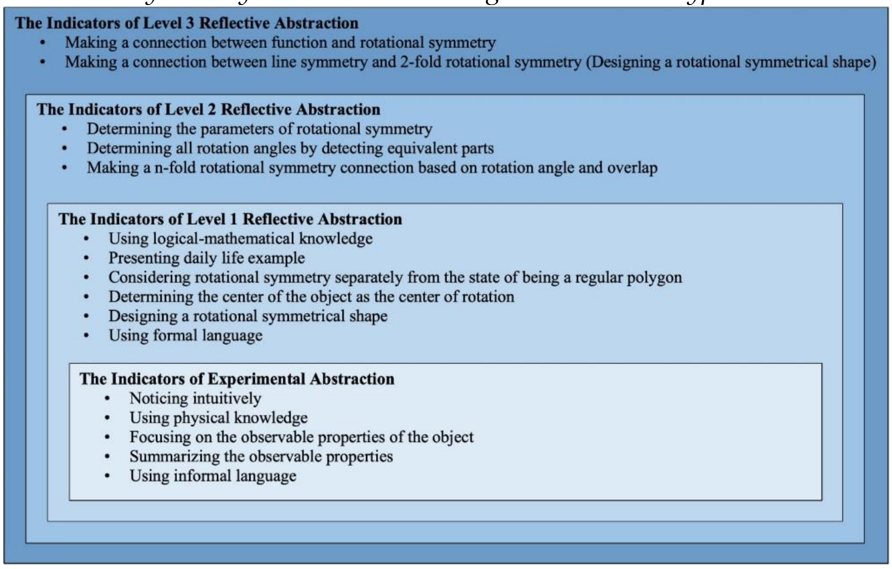
Figure 14
Rotational symmetrical shape designed by Ahu



3.3. Results Related to Indicators according to Levels of Abstraction

The assignment of the behaviors that emerged in the abstraction processes of the concept of rotational symmetry was carried out by considering the hierarchical relationship according to the levels in Piaget's abstraction schema. These behaviors were determined as indicators according to the abstraction type and levels of the concept of rotational symmetry. The indicators according to the abstraction type and levels are summarized in Figure 15.

Figure 15
Rotational symmetry indicators according to abstraction type and levels



The indicators of experimental abstraction were identified as students' intuitively noticing the rotational symmetricity in the daily life examples, using only physical knowledge, focusing on the observable properties of the object, summarizing the observable properties, and using informal language about rotational symmetry. In terms of rotational symmetry, these indicators fall short of providing full conceptual understanding. Furthermore, they perform the function of a mediator in providing an indirect connection to conceptual understanding.

As indicators of reflective abstraction level 1, students were required to make connections based on their logical and mathematical knowledge, present examples of rotational symmetry in daily life, and consider rotational symmetry separately from regular polygons, determine the center of rotation, design a rotational symmetrical shape, and use formal language about rotational symmetry. Providing conceptual understanding of rotational symmetry, these indicators enabled actions independent of current environment and object properties. There is, however, a need to establish a connection between these actions.

Reflective abstraction indicators at level 2 include students' ability to determine rotational symmetry parameters from daily life examples, determine all rotation angles from the equivalent parts of rotational symmetrical shapes, and establish the relationship between rotation angles and n-fold rotational symmetrical shapes. This level of indicators was determined by re-entering indicators at the previous level into the reflective abstraction process. In addition, it can be said that it is an indicator of being able to distinguish thinking and providing conceptual understanding of rotational symmetry.

Based on the rotational symmetrical shape they drew from a daily life example, students were able to establish a connection between function and rotational symmetry, and establish a connection between line symmetry and twofold rotational symmetry, which were indicators of reflective abstraction level 3. Considering the hierarchical relationship, it can be stated that these indicators, together with the other indicators, are necessary and sufficient in establishing the most inclusive and general connections for full conceptual understanding of rotational symmetry.

4. Discussion and Conclusion

According to this study, none of the students in the participants had formal knowledge of rotational symmetry prior to the pre-clinical interviews, and they could only notice rotational symmetry intuitively in the examples they examined in daily life. When students were asked what they thought of rotational symmetry in the clinical interview that followed, it was striking that two of them were able to define it informally, albeit incorrectly, and provide examples from their daily lives, whereas one student was unable to express an opinion, stating that she had never heard of rotational symmetry before. Among the three students, two were at the experimental abstraction level regarding rotational symmetry, while the student who could not express an opinion was neither at the reflective abstraction level nor at the experimental abstraction level. As a result, none of the students were at a reflective abstraction level, since rotational symmetry is only one of the 12th grade subjects in the secondary school curriculum, and that section only deals with symmetry with respect to the point, which is a special case of rotational symmetry. In Köse (2012), symmetry relative to the point is not fully established in students, and Durmuş (2017) indicates that students are deficient in symmetry when it comes to symmetry relative to the point. The results of this study are supported by both of these studies. When the students were asked to photograph and send the objects that captured their attention as rotational symmetry in their environment, the fact that the student, who could not give any examples of rotational symmetry from daily life at the very beginning of the teaching experiment, was able to send photographs containing rotational symmetrical objects under a wide variety of themes by focusing on both nature, close environment and ornaments was an important result in terms of increasing the awareness of rotational symmetry in daily life.

Two students were able to analyze the polygons by reasoning after being asked whether they were rotationally symmetric. As a student examined the polygons, one student was able to divide them into congruent parts based on their center points and proceeded through these congruent parts. After passing through these congruent parts, the student was able to determine that all polygons are rotationally symmetrical; however, because the angles of rotation were not considered, the student had some shortcoming. Upon examining rotational symmetry in daily life examples, the student realized that angles of rotation should also be considered and corrected his deficiencies in polygons. The other student, who was capable of making an analysis, analyzed regular polygons and non-regular polygons with the idea that regular polygons are rotationally symmetric polygons. The student concluded that rotational symmetry is also possible for shapes that are not regular polygons, once he started analyzing daily life examples. The student who could not analyze anything for the first time was able to start reasoning about being rotationally symmetric after looking at examples from daily life. It was concluded that the student was motivated by everyday life examples (Albayrak et al., 2017). Another important finding in this study is that using daily life examples while examining rotational symmetry strengthens students' ability to make connections between concepts by strengthening their abstraction mechanisms of the concept of rotational symmetry (Doruk & Çiltaş, 2020). According to Marchis (2009), daily life examples are effective for discovering symmetry and are parallel to the results of his study.

Two of the students did not reach the highest level of reflective abstraction based on the indicators related to abstraction type and level. In regard to the hierarchical relationship between the determined indicators, it was determined that the reason for this situation was the inability to establish a strong enough connection between the previously known concept of rotational symmetry and rotation symmetry. The only student who reached level 3 of reflective abstraction was able to realise all the relationships between concepts. The student demonstrated the connection between rotational symmetry and function by determining the origin of a coordinate plane as the center of rotation, drawing a four-fold rotational symmetrical shape composed of rectangles and treating their corner points as functions under rotational transformation. Additionally, the student was able to use symmetry with respect to the line to establish the connection between 2-fold rotational symmetry and rotational symmetrical shapes. Again, in the study conducted by Wafiqoh and Kusumah (2019) within the framework of the reflective abstraction levels presented by Cifarelli (1988), the reason why students were unable to attain the highest level of reflective abstraction related to mathematical problem solving was the same as the lack of connection between prior knowledge and newly encountered concepts.

Before the teaching experiment related to the concept of rotational symmetry, two students were at the experimental abstraction level and progressed to reflective abstraction levels 2 and 3, whereas the other student could only reach reflective abstraction level 1 as he/she was not at any abstraction level before the teaching experiment. This student had the most trouble establishing relationships between concepts, discovering the parameters related to concepts and abstractions among the students. Based on this analysis, it was concluded that all students improved in their abstraction of rotational symmetry compared to the level they were at before the teaching experiment, and that their progress was related to their level before the teaching experiment. Goodson-Espy (1998) aimed to reveal students' understanding of the concept of linear inequality through problem solving according to Cifarelli's (1988) reflective abstraction levels in his study. In line with the current study's findings, Goodson-Epsy (1998) concluded that students with a higher level of reflective abstraction had less difficulty in establishing relationships between concepts and students with a lower level of reflective abstraction had more difficulty or were not successful.

Students were asked to design rotational symmetrical shapes as part of their last clinical interview in order to better internalize the knowledge they gained through abstraction about rotational symmetry. The design of the student who advanced to level 1 of reflective abstraction contained only line segments, whereas the designs of the students who advanced to levels 2 and 3 of reflective abstraction contained curves as well as line segments. Students followed different paths and used different geometric objects in their designs, but the fact that they designed 4-fold rotational symmetrical shapes was an interesting finding. Despite all students designing a 4-fold rotational symmetrical shape, it was determined that the level of abstraction reached and the shape they designed increased in direct proportion to the level of abstraction. A student who reached level 1 of reflective abstraction had the simplest design, while a student who reached level 3 of reflective abstraction had the most complex design. Students preferred 4-fold rotational symmetrical shapes because they were easier to draw and suitable for prototyping since rotation angles of 90° are relatively easy to draw. According to research, students' rotational symmetrical shapes help them make sense of their knowledge. The study conducted by Aktaş (2015) aimed to enhance students' symmetry learning through computer animations and exercises. According to the results, students' conceptual understanding of symmetry was enhanced by the ornaments they drew after the teaching activities had been completed and the ornaments they made using various types of symmetry.

In addition to polygons and symmetrical shapes, other content was evaluated, such as making connections between shapes. In addition to examining daily life examples and drawing, the teaching experiment included other activities that were effective in improving students' individual knowledge and understanding of rotational symmetry by exploring the relationships between concepts in a systematic manner. This result supports Özaltun-Çelik's (2018) conclusion that students must have the opportunity to learn by doing, experiencing and discovering through the teaching sequence activities designed to enable them to perform reflective abstraction by supporting their conceptual learning about quadratic functions. In the same vein, Camci (2018) concluded that the content of the classroom-based teaching experiment he designed within the framework of the hypothetical learning trajectory revealed the mechanisms of students through the use of Piaget's classical abstraction structure, which supported the abstraction process and development of all students participating in it.

5. Recommendations

Considering the research results, following recommendations are offered:

- The use of daily life examples in teaching rotational symmetry is recommended more frequently since it strengthens students' knowledge and understanding. To do this, mathematics teachers can receive in-service training on the importance of using real-life examples in their teaching activities.

- In light of the fact that drawing rotational symmetrical shape designs using the knowledge they have acquired about rotational symmetry assists students in internalizing the concept, it is recommended that more drawing activities be used when designing teaching activities.
- Rotational symmetry was abstracted successfully through one-to-one teaching. Students' conceptual understanding can be strengthened by activities that support Piaget's level of reflective abstraction and that require them to make connections at a variety of levels.
- Students did not have a reflective abstraction about rotational symmetry prior to the teaching experiment. When updating/revising mathematics curricula, a comprehensive section on rotational symmetry should be included.
- An abstract representation of rotational symmetry is presented in this study. It is possible to plan experiments that include more than one type of symmetry, such as line symmetry, translational symmetry, rotation symmetry, etc., and that enable the students to connect them.

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