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Research Article

Incorrect theorems and proofs: An analysis of pre-service mathematics teachers' proof evaluation skills

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Proof facilitates conceptual and meaningful learning in mathematics education rather than rote memorization. In this study, incorrect theorems and proofs are used to assess secondary school pre-service mathematics teachers' proof assessing skills. Using the case study method, the study is conducted on pre-service mathematics teachers studying at the Department of Mathematics Education. There were eight pre-service mathematics teachers selected from each grade, resulting in 32 participants in total. A semi-structured proof form containing 13 questions was used to collect data, which was analyzed using content analysis. As the analysis reveals, pre-service mathematics teachers are highly likely to make incorrect decisions regarding theorems and proofs, and the margin of error is unaffected by grade level. Moreover, pre-service mathematics teachers tend to use proving terms incorrectly and, at times, are unable to differentiate between terms that are commonly used in proving. The pre-service mathematics teachers are believed to have learned proofs by rote rather than understanding how proofs work. With the help of interviews and tests created for different proof methods, it has been suggested that pre-service mathematics teachers should be tested on their proof evaluation skills in more detail.

Keywords: Mathematics education; Proof assessment; Mathematical proof; Pre-service mathematics teachers

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1. Introduction

The quest for truth has been part of humanity's history for centuries. Having the right knowledge helps us solve current problems, as well as anticipate and solve potential future problems. Most educational programs adhere to the constructivist approach, which emphasizes the importance of true knowledge. According to this approach, new knowledge is constructed on the basis of existing knowledge (Karadag et. al. 2008). The constructivist approach places a high value on the ability to separate correct from incorrect information. According to the constructivist approach, information must be evaluated on the basis of its truth and the reasoning supporting that evaluation. There are different methods for measuring truth in social disciplines. In mathematics, this method is known as mathematical proof (Aksoy & Narlı, 2019). According to Yıldırım (2014), mathematical proof is the effort to prove the truth (or incorrectness) of a thesis or judgement.

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Weber (2005) describes it as a mathematical activity in which the person proving the result presents old information such as definitions, axioms, and assumptions, and the inference rules are used until the desired result is achieved by applying theorems and remembering previous information. A number of mathematics educators have argued that proofs should not only be used to convince students that mathematical statements are true, but also to facilitate communication, explain concepts, and increase understanding in the classroom (De Villiers, 1990; Hanna, 1991; Hersh, 1993; Knuth, 2002; Weber, 2010). The purpose of proof is twofold. Firstly, we show that assumptions can lead to conclusions by following logical steps; secondly, we explain the whys and hows behind the assumptions (Tall, 1998). Theorems are proved in this way not only to show the truth of a theorem, but also to illustrate its validity.

Proof is considered essential to mathematics education by many mathematicians and educators (Senk et.al. 2009, as cited in Bahtiyari 2010; Turner 2010). Undergraduate students tend to encounter serious difficulties when attempting to engage with proof in the intended manner according to Almeida (2000). Students learn mathematics concepts better when they have the opportunity to engage in mathematical proof (Hersh, 1993). Mathematical information is established, developed, and transferred through proof (Stylianides, 2007). In proving mathematical knowledge (Hanna, 1991; Kitcher, 1984), one finds meaning behind the mathematicians' actions. Students benefit from mathematical proof by developing their critical thinking and problem-solving abilities (Fawcett, 1938; Rav, 1999). Teachers must ensure that their students hone this skill by ensuring that they are exposed to mathematical proof. A study by Peterson et al. (1989) supports this notion by concluding that teachers' perspectives and approaches to a subject affect students' achievement. Researchers emphasize the importance of mathematical proof, but students and pre-service mathematics teachers from all levels of education seem to have difficulty understanding it (Almeida, 2000; Arslan, 2007; Arslan & Yıldız, 2010; Aydogdu et al., 2003; Coskun, 2009).

The topic of mathematical proof is a subject of ongoing research in mathematics education, and a simple search can reveal that a significant amount of research has been conducted on it over the years. The study of proof can be divided into two main categories: producing proofs and evaluating (understanding, comprehending) proofs. Mejía-Ramos and Inglis (2009) found that most studies on proofs investigated students' proof construction, while small groups of studies concentrated on students' proof reading. Participants are typically asked to prove statements given to them, and various analytical methods are used to evaluate the quality of these proofs and the participants' level of proving ability. There are many studies on providing the proofs by the participants (e.g., Hoyles & Healy 2007; Sevgi & Kartalcı, 2021; Weber, 2001, 2005). The studies in the 'evaluating proofs' category ask participants to judge what method(s) are used and whether the methods are correct. The purpose of these studies is to have participants evaluate existing proofs. There was less attention paid to the comprehension of written proofs (Almeida, 2000, 2003; Mejia-Ramos & Inglis, 2009). There are also some other studies on this topic (Doruk & Kaplan, 2013; Guler & Emekci, 2016; Imoğlu & Yontar Togrol, 2015; Rapke & Allan, 2016) are some names who have conducted such research.

As teachers are responsible for ensuring their students gain mathematical proof abilities (Arslan 2007; Carpenter 1989; Peterson et al., 1989), it is vital for pre-service mathematics teachers to be able to prove and evaluate proof. The research shows that the pre-service mathematics teachers often struggles with proof (Almeida, 2000; Arslan, 2007; Arslan & Yıldız, 2010; Aydogdu et al., 2003). To produce proofs, pre-service mathematics teachers must be able to comprehend a mathematical text, follow the steps of a preexisting proof, and evaluate it (Powers et al., 2010). In spite of the fact that students in mathematics education departments are taught how to produce proof (Cusi & Malara, 2007; Ko & Knuth, 2009), give counter examples (Riley, 2003; Zaslavsky & Peled, 1996), and evaluate proof (Guler & Emekci, 2016; Knuth, 2002; Uygan et al., 2014), they were not performing well. A study conducted by Almeida in 2000 assessed undergraduates' ability to evaluate proof accuracy. An evaluation of their understanding of proof and comparison with that of mathematicians was conducted using a questionnaire. According to Almeida (2000), students understand proof differently than mathematicians.

This study examines how pre-service mathematics teachers evaluate incorrect theorems and incorrect proofs as a way to assess their proof evaluating abilities. Essentially, this work defines *proof evaluating abilities* as understanding proofs and all their steps, as well as recognizing mistakes and their underlying reasons. It is our belief that the results produced in this study will add to the body of literature on evaluating proof. This study aims to assess the knowledge, ability, and shortcomings of pre-service mathematics teachers regarding understanding theorems and proofs, as well as the areas they tend to focus on when evaluating them, using incorrect theorems and proofs.

2. Method

The case study method was used to assess pre-service mathematics teachers' knowledge and ability to evaluate proof and their deficiencies in these areas. Using case studies, researchers can examine an aspect of a subject in depth and interpret the environment and events surrounding it (Yıldırım & Şimşek, 2011). Using multiple data gathering techniques for a wide variety of data, case studies focus on the special aspect of a problem (Çepni, 2007). It is important for case studies to specify a question of cause and process, investigate a contemporary phenomenon rather than a historical phenomenon, and to meet some criteria, such as examining the phenomenon in context and in detail (Yin, 2017).

2.1. Participants

The study involved 32 pre-service mathematics teachers attending a state university in the Aegean region of Turkey during the 2019-2020 academic year. Purposive sampling methods such as convenience sampling and volunteerism were used to select the participants. Eight participants were selected from each grade level to determine whether there is a difference between different grade levels. Each participant has attended a course in their first year called Abstract Mathematics, which covers logic, proof methods, and applications. The students in further grade levels take algebra and topology courses that require proof knowledge.

2.2. Data Collection Tools

To assess the pre-service mathematics teachers' ability to evaluate incorrect theorems and proofs, a questionnaire consisting of thirteen open-ended questions was prepared. Experts assisted with selecting questions from Velleman's (2006) book. Due to the fact that participants came from different classes, it was important that the questions were suitable, understandable, and solvable. We preferred questions that contained general mathematical proofs rather than specific subjects like algebra, analysis or topology, and that did not focus on one type of error. Additionally, the questions were written in a clear, understandable manner. Participants were required to answer open ended questions such as "correct theorem/incorrect theorem, because...," or "correct proof/incorrect proof, because...". There are some questions in the study that have correct theorems and/or proofs, even though the majority of the questions consist of incorrect theorems and proofs. For example, the fifth question has the correct theorem, and the seventh and ninth questions have both the correct theorem and proof. As a result, participants will not be tempted to find mistakes randomly because they know they exist. By including correct statements, we prevented participants from searching for mistakes in the questions rather than contemplating the given theorems and proofs. Researcher prepared an answer key following the preparation of the open-ended questionnaire. The answer key was prepared taking into account the possibility that participants might approach the questions differently. Participants were all present at the same time during the data collection session. Despite no time limit, most students found two hours to be sufficient, even though there was no pressure to finish.

2.3. Data Analysis

Content analysis was used to interpret the pre-service mathematics teachers' answers. A concept, theme, and code can be determined in advance or found and explained during the analysis process (Sakarya & Zahal, 2020) when expressing data using various concepts and relations. In terms of proof methods, given theorems, and reasons why they were unable to find mistakes in the proofs, researchers examined the answers given by the pre-service mathematics teachers. We considered correct answers that noted the mistakes, followed the steps of operation, and checked whether the given theorem was correct by following the proof method. Answers that contain logical arguments but cannot fully justify the given answer are categorized as partially correct and the others as incorrect. Through percentages and frequencies, "correct theorems/incorrect theorems" and "correct proofs/incorrect proofs" were analyzed. A table containing the "correct/incorrect/partially correct" answers for both theorems and proofs, as well as the reasons given for their answers, was created from the data analysis of each participant's questionnaire. In addition, titles and charts were commonly used elements within the answers. As part of the analysis process, the researchers met regularly to compare notes, discuss, and make adjustments. In their analysis, the researchers' coefficient of concordance fluctuates between 83 and 95%.

3. Findings

Answers to the research questionnaire provided by the pre-service mathematics teachers are included in this section. Answers to the questions about the correctness or incorrectness of theorems and proofs are presented in Table 1 as correct/incorrect/blank in terms of percentage and number.

Table 1
Participant responses about the correctness or incorrectness of theorems and proofs

	f			%				*f*			%		
Theorems	C	I	B	C	I	B	Proofs	C	I	B	C	I	B
Theorem 1	9	20	3	28.13	62.50	9.38	Proof 1	9	21	2	28.13	65.63	6.25
Theorem 2	26	4	2	81.25	12.50	6.25	Proof 2	28	4	0	87.5	12.5	0
Theorem 3	6	25	1	18.75	78.13	3.13	Proof 3	6	25	1	18.75	78.12	3.13
Theorem 4	29	2	1	90.63	6.25	3.13	Proof 4	29	1	2	90.63	3.12	6.25
Theorem 5*	32	0	0	100.00	0.00	0.00	Proof 5	31	1	0	96.88	3.12	0
Theorem 6	26	6	0	81.25	18.75	0.00	Proof 6	24	8	0	75	25	0
Theorem 7*	27	3	2	84.38	9.38	6.25	Proof 7*	25	5	2	78.12	15.63	6.25
Theorem 8	10	18	4	31.25	56.25	12.50	Proof 8	9	18	5	28.12	56.25	15.63
Theorem 9*	29	2	1	90.63	6.25	3.13	Proof 9*	28	3	1	87.5	9.37	3.13
Theorem 10	7	20	5	21.88	62.50	15.63	Proof 10	6	20	6	18.75	62.5	18.75
Theorem 11	10	6	16	31.25	18.75	50.00	Proof 11	8	8	16	25	25	50
Theorem 12	23	6	3	71.88	18.75	9.38	Proof 12	22	7	3	68.75	21.87	9.38
Theorem 13	15	14	3	46.88	43.75	9.38	Proof 13	15	14	3	46.87	43.75	9.38
TOTAL	249	126	41	59.85	30.29	9.85	TOTAL	240	135	41	57.69	32.45	9.85

Note. C: Correct; I: Incorrect; B: Blank; * Correct theorems and proofs

Table 1 shows that participants had the least success finding mistakes in the 3rd and 10th theorems and proofs. Although the theorems and proofs were incorrect, more than half of the pre-service mathematics teachers thought they were correct. According to Table 1, the topics of algebra and sets were the subjects with the highest percentage of correct answers for the theorems and proofs 4th and 5th. According to Table 1, students' responses to the correctness of the theorems and proofs are mostly consistent.

Table 2 presents the pre-service teachers' reasons as to why a theorem or proof was correct or incorrect, categorized as correct/incorrect/partially correct/blank. The answers classified as partially correct include logical arguments but are unable to fully justify the given answer.

Table 2
Participant responses about the reasons for their answers

	Reasons																
	f				%					f				%			
Theorems	C	I	B	P	C	I	B	P	Proofs	C	I	B	P	C	I	B	P
Theorem1	5	20	5	2	15.63	62.50	15.63	6.24	Proof 1	4	21	6	1	12.50	65.63	18.74	3.13
Theorem 2	18	4	5	5	56.24	12.50	15.63	15.63	Proof 2	16	4	5	7	50.00	12.50	15.63	21.87
Theorem 3	3	22	4	3	9.38	68.74	12.50	9.38	Proof 3	4	22	4	2	12.50	68.75	12.50	6.25
Theorem 4	23	2	5	2	71.88	6.24	15.63	6.25	Proof 4	22	1	7	2	68.75	3.13	21.87	6.25
Theorem 5*	27	1	2	2	84.37	3.13	6.25	6.25	Proof 5	25	1	3	3	78.13	3.13	9.37	9.37
Theorem 6	19	7	3	3	59.37	21.87	9.38	9.38	Proof 6	14	9	5	4	43.74	28.13	15.63	12.50
Theorem 7*	20	3	5	4	62.50	9.38	15.62	12.50	Proof 7*	21	6	2	3	65.63	18.75	6.25	9.37
Theorem 8	4	16	8	4	12.50	50.00	25.00	12.50	Proof 8	6	14	10	2	18.75	43.75	31.25	6.25
Theorem 9*	21	2	4	5	65.63	6.24	12.50	15.63	Proof 9*	20	3	5	4	62.50	9.37	15.63	12.50
Theorem 10	3	18	9	2	9.38	56.25	28.12	6.25	Proof 10	3	21	7	1	9.37	65.63	21.87	3.13
Theorem 11	5	6	19	2	15.63	18.75	59.37	6.25	Proof 11	4	8	18	2	12.50	25.00	56.25	6.25
Theorem 12	16	5	7	4	50.00	15.63	21.87	12.50	Proof 12	14	6	7	5	43.75	18.75	21.87	15.63
Theorem 13	12	11	6	3	37.50	34.37	18.75	9.38	Proof 13	10	13	6	3	31.25	40.63	18.75	9.37
TOTAL	176	117	82	41	42.30	28.12	19.71	9.86	TOTAL	163	129	85	39	39.18	31.01	20.43	9.38

Note. C: Correct; I: Incorrect; B: Blank; P: Partially correct.

Students are categorized in Table 2 by the reasons they gave for their responses regarding the correctness of theorems and proofs. The justifications in Table 2 have more unanswered items than those in Table 1. Only half of the participants were able to explain their answers sufficiently. In examining the pre-service mathematics teachers' responses, it was found that the answers provided were quite limited in variety. Both tables show low rates of correct answers and superficial explanations for theorems and proofs errors.

It can be seen from Table 2 that the number of reasons correctly provided by the participants for the 3rd and 10th theorems and proofs is also the lowest. Table 2 also shows that most correct explanations they provide are based on 4th and 5th questions. It is worth noting that algebra and sets are both subjects with a lot of proofs and theorems. These factors may contribute to the participants' familiarity with proofs in these areas, and their ability to spot mistakes. In these cases, participants were able to identify whether these questions were correct or not, but they were not able to sufficiently explain and give reasons why these proofs were correct. Proof 11 and theorem 11 are mostly blank for the correctness and reasons.

Figure 1 shows the most common situations encountered in the answers to each question.

Figure 1
Student Attempts/Outcomes on the Correctness of Proofs

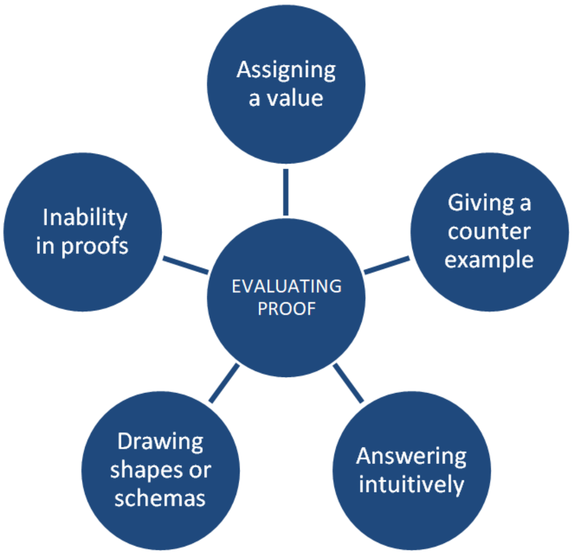

Figure 1 illustrates how pre-service mathematics teachers evaluate theorems and proofs by assigning a value, giving a counterexample, answering intuitively, and drawing shapes or schemas. Furthermore, the pre-service mathematics teachers' answers put forth that they have gaps in their knowledge of proof methods, or in particular subject areas, which prevent them from successfully evaluating theorems and proofs. Following are examples of common types of answers participants gave for each question. Table 3 includes a response of a pre-service teacher regarding the first question in data collection tool. As in Example 1, most of the responses the pre-service mathematics teachers provided to this question were incorrect. Due to their inability to find the mistake, they assumed the proof was correct, and therefore the theorem was correct. To crosscheck the theorem, most participants assigned a value, as shown in Example 2. In this question, there were no significant differences between grade levels.

Table 3
Examples of Responses for Question 1

Theorem 1: $\exists\, x \in R, \forall\, y \in R\ (xy^2 = y - x)$ Proof: Let $x = \frac{y}{y^2+1}$. Then $y - x = y - \frac{y}{y^2+1} = \frac{y^3}{y^2+1} = \frac{y}{y^2+1} \cdot y^2 = xy^2$ (Velleman, 2006)	
Example 1: Making misjudgement Theorem: Correct/Incorrect (no answer given) Reason; $xy^2 + x = y$ $x(y^2+1) = y$ $x = \frac{y}{y^2+1}$ $\forall y$ satisfy this equation, an x can be found to satisfy this equation. Proof: Correct/Incorrect (no answer given) Reason; There are no errors in the mathematical operations.	*Example 2: Examining the correctness of theorem and proof by assigning values* Theorem: Correct/Incorrect (no answer given) Reason; Doesn't satisfy for (1,2) $(1, 2^2) \neq (2, -1)$ Proof: Correct/Incorrect (no answer given) Reason; (no answer given)

Table 4
Examples of Responses for Question 2

Theorem 2: For all real numbers x and y, $x^2 + xy - 2y^2 = 0$. Proof: Let x and y be equal to an arbitrary real number. Then, $x^2 + xy - 2y^2 = r^2 + rr - 2r^2 = 0$. Since both x and y were arbitrary, for all x and y real numbers $x^2 + xy - 2y^2 = 0$ (Velleman, 2006).	
Example 3: Examining the correctness of theorem and proof by assigning values Theorem: Correct/*Incorrect* Reason; $x = 0$ For $y = 1$ $0^2 + 0 - 2 \neq 0$ Proof: Correct/*Incorrect* Reason; If we let x and y be equal to r, than rr is found. But the theorem suggests that for all real numbers, so $-rr$ wasn't taken into consideration.	*Example 4: Making a correct assessment* Theorem: Correct/*Incorrect* Reason; $x = \sqrt{2}$ $(\sqrt{2}^2) + \sqrt{2}.\sqrt{3} - 2\sqrt{3}^2) \neq 0$ Theorem is set wrong. Let $y = \sqrt{3}$ Proof: Correct/*Incorrect* Reason; An arbitrary number r is chosen for x and y, all real numbers weren't considered. It doesn't mean it is true for all numbers.

In response to the second question given in Table 4, the pre-service mathematics teachers mainly followed Examples 3 and 4. There was no problem finding the mistake in the proof, with most participants noting that the theorem was incorrect.

Table 5
Examples of Responses for Question 3

Theorem 3: Let x and y be real numbers and $x + y = 10$. Then $x \neq 3$ and $y \neq 8$. Proof: Let's consider the conclusion of the theorem is incorrect. Than $x = 3$ and $y = 8$. But then $x + y = 11$. It contradicts the given result $x + y = 10$. Hence, the result must be wrong (Velleman, 2006)	
Example 5: Lack of knowledge on proving methods Theorem: *Correct*/Incorrect Reason; The proof was conducted by contradiction. When the result was wrong, the hypothesis was also wrong. Then the hypothesis is true. Proof: *Correct*/Incorrect Reason; (no answer given)	*Example 6: Making a misjudgement* Theorem: *Correct*/Incorrect Reason; When $x = 3$ and $y = 8$ simultaneously, as the theorem says, the result won't be 10. Also, since it doesn't say for all x and y, it is valid for some specific x and y. Proof: *Correct*/Incorrect Reason; Proof is done by contradiction. In this method the inverse of the theorem assumed to be correct and then contradiction is found. So the correctness of the statement is proven.

The third question (see Table 5) was answered incorrectly by most participants. This was largely due to an inadequate understanding of proof methods, similar to Example 5. Participants could not find any errors in the proof of Example 6 because they assigned values to determine if the theorem was correct. Both examples demonstrate participants' lack of understanding of simple mathematical logic. Consequently, the majority incorrectly assumed the proof was true.

Table 6
Examples of Responses for Question 4

Theorem 4: Suppose that $A \subset C, B \subset C$ and $x \in A$. Then $x \in B$. Proof: Suppose that $x \notin B$. Since $x \in A$ and $A \subset C, x \in C$. Since $x \notin B$ and $B \subset C, x \notin C$. But now we have proven both $x \in C$ and $x \notin C$, so we have reached a contradiction. Therefore $x \in B$ (Velleman, 2006).
Example 7: Making sense of the proof by drawing a diagram-shape Theorem: Correct/Incorrect (no answer given) Reason; I'll show that it is incorrect using venn diagram. If $A \cap B = \varnothing$, an element x of A cannot be in B. Proof: Correct/Incorrect (no answer given) Reason; This is a wrong sentence. $B \subset C$. If an element not in B, it can be in $C \backslash B$.

As shown in Table 6, it can be seen from example 7 that nearly all of the participants answered the fourth question correctly. There are likely two reasons for this high success rate: pre-service mathematics teachers know enough about sets to understand the question, and the question lends itself to interpretation using shapes.

Table 7
Examples of Responses for Question 5

Theorem 5: Suppose x is a real number and $x \neq 4$. If $\frac{2x-5}{x-4} = 3$ then $x = 7$. Proof: Suppose $x = 7$. Then $\frac{2x-5}{x-4} = \frac{2.(7)-5}{7-4} = \frac{9}{3} = 3$. Therefore if $\frac{2x-5}{x-4} = 3$ then $x = 7$. (Velleman, 2006)	
Example 8: Declaring that the proof method is incorrect Theorem: *Correct*/Incorrect Reason; They gave the proposition clearly by removing the case of making the denominator 0. Proof: Correct/*Incorrect* Reason; Proof can't be done by just substituting numbers directly. We must consider that there might be some other x which doesn't satisfy the condition.	*Example 9: Making a Correct Assessment* Theorem: *Correct*/Incorrect Reason; There is no wrong statement here. Proof: Correct/*Incorrect* Reason; We can't do the proof by substituting. It would be more accurate to show that x is equal to 7 by solving the equation.

The fifth question (see Table 7) was answered correctly by almost all participants. They were able to spot the mistake within the question based on Examples 8 and 9, indicating that the method used was not an actual proof method.

As a result of utilizing the information in the theorem presented in Table 8, the pre-service mathematics teachers were able to correctly evaluate the proof in Question 6. Participants in the second and third years answered the question incorrectly at a lower rate than those in the first year. Using the information provided in the theorem, they determined the mistake in the proof. It is also possible that the high rate of correct answers is due to participants' knowledge of algebraic operations.

Table 8
Example of Responses for Question 6

Theorem 6: Suppose that x and y are real numbers and $x \neq 3$. If $x^2y = 9y$ then $y = 0$.
Proof: Suppose that $x^2y = 9y$. Then $(x^2 - 9)y = 0$. Since $x \neq 3, x^2 \neq 9$, so $x^2 - 9 \neq 0$. Therefore we can devide both sides of the equation $(x^2 - 9) = 0$ by $x^2 - 9$, which leads to the conclusion that $y = 0$. Thus, if $x^2y = 9y$ then $y = 0$. (Velleman, 2006)

Example 10: Recognising the incompleteness in the theorem ($x = -3$)
Theorem: Correct/*Incorrect*
Reason; (no answer given)
Proof: Correct/*Incorrect*
Reason; $x = -3$ or $y = 0$. $x^2y - 9y = 0$
$y(x^2 - 9) = 0,\ x = 3, x = -3$

A number of participants answered the 7th question (see Table 9) intuitively without providing an explanation, as shown in Example 11. Due to the subject matter being set, the question being suitable for using shapes and schemas for visualisation (as in Example 12), and both the proof and the theorem being correct, there may be a high rate of correct answers.

Table 9
Examples of Responses for Question 7

Theorem 7: For any sets A, B and C, if $A\backslash B \subset C$ and $A \not\subset C$ then $A \cap B \neq \emptyset$
Proof: Since $A \not\subset C$, we can choose some x such that $x \in A$ and $x \notin C$. Since $x \notin C$ and $A\backslash B \subset C$, $x \notin A\backslash B$. Therefore either $x \notin A$ or $x \in B$. But we already know that $x \in A$, so it follows that $x \in B$. Since $x \in A$ and $x \in B$, $x \in A \cap B$. Therefore $A \cap B \neq \emptyset$. (Velleman, 2006)

Example 11: Responding intuitively
Theorem: *Correct*/Incorrect
Reason; I think it is correct.

Example 12: Making sense of the theorem by drawing a diagram-shape
Theorem: *Correct*/Incorrect
Reason; Some elements of A must be in C, so, it must be
$A \cap B \neq \emptyset$.

Table 10
Examples of Responses for Question 8

Theorem 8: There are irrational numbers a and b such that a^b is rational

Proof: Either $\sqrt{2}^{\sqrt{2}}$ is rational or irrational.

Case 1: $\sqrt{2}^{\sqrt{2}}$ is rational. Let a=b=$\sqrt{2}$. Then a and b are irrational, and $a^b = \sqrt{2}^{\sqrt{2}}$, which we are assuming in this case is rational.

Case 2: $\sqrt{2}^{\sqrt{2}}$ is irrational. Let a=$\sqrt{2}^{\sqrt{2}}$ and b=$\sqrt{2}$. Then a is irrational by assumption and b is also irrational. Also,
$a^b = (\sqrt{2})\sqrt{2}^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}.\sqrt{2}} = \sqrt{2}^2$ =2 which is rational. (Velleman, 2006)

Example 13: Confusion due to the double case given in the proof
Theorem: Correct/Incorrect (no answer given)
Reason; I don't know such a theorem.
Proof: Correct/*Incorrect*
Reason; In case 1, $(\sqrt{2})^{\sqrt{2}}$ considered to be rational. But in case 2, it says a=$\sqrt{2}^{\sqrt{2}}$ is irrational. Two cases contradict each other.

Example 14: Lack of subject knowledge
Theorem: *Correct*/Incorrect
Reason; Looking at its proof, it looks like a logical theorem.
Proof: *Correct*/Incorrect
Reason; Two different cases were examined. It is true because there is at least one satisfying condition.

Most participants answered the eighth question presented in Table 10 incorrectly. They did not have any prior knowledge of the theorem or its proof, and they had to check two different situations for correctness. The lack of knowledge and the double condition have led to confusion, as shown in examples 13 and 14.

Table 11
Examples of Responses for Question 9

Theorem 9: For any sets A, B, C and D, if $AxB \subset CxD$ then $A \subset C$ ve $B \subset D$. Proof: Suppose $AxB \subset CxD$. Let a be an arbitrary element of A and b be an arbitrary element of B. Then $(a, b) \in AxB$, so since $AxB \subset CxD$, $(a, b) \in CxD$. Therefore $a \in C$ and $b \in D$. Since a and b were arbitrary elements of A and B, respectively, this shows that , $A \subset C$ and $B \subset D$. (Velleman, 2006)	
Example 15: Responding intuitively Theorem: *Correct*/Incorrect Reason; Logical. We proved this in class earlier. Proof: *Correct*/Incorrect Reason; Proof is correct. We did this before.	*Example 16: Responding intuitively* Theorem: *Correct*/Incorrect Reason; (no answer given) Proof: *Correct*/Incorrect Reason; (no answer given)

There was a high percentage of correct answers for the 9th question among pre-service mathematics teachers. Students are mostly able to remember the proof and the theorem due to their prior knowledge, as shown in Example 15. Similarly to Example 16, some students gave the correct answer but did not explain why. Using rote learning to learn proofs may lead to a lack of reasoning.

Table 12
Examples of Responses for Question 10

Theorem 10: Suppose R is a relation on A. If R is symmetric and transitive, then R is reflective. Proof: Let x be an arbitrary element of A and y be arbitrary element of A satisfying xRy. Since R is symmetric, yRx. But by transition property we get xRx. Since x is arbitrary we showed for $\forall x \in A(xRx)$. So R is reflective. (Velleman, 2006)	
Example 17: Lack of subject knowledge Theorem: *Correct*/Incorrect Reason; If a relation on A provides the property of symmetry and transitivity, it also provides the property of reflexivity. Proof: *Correct*/Incorrect Reason; An arbitrary element is chosen, using symmetry and transitivity, it tried to reach the desired result by using induction.	*Example 18: Explaining by using counterexample.* Theorem: Correct/*Incorrect* Reason; (no answer given) Proof: Correct/*Incorrect* Reason; Let A be $A = \{a, b, c\}. \{(a, b), (b, a), (a, a), (b, b)\}$ Since there is no (c, c), it is not reflexive.

In response to the 10th question shown in Table 12, most pre-service mathematics teachers provided incorrect answers. It is probably a result of insufficient knowledge of the subject of relations, as well as rote learning. When asked to explain their answers, participants have also struggled. Most participants who reached the correct answer used counter examples to explain their answers, as shown in Example 18.

Table 13

Example of Responses for Question 11

Theorem 11: Suppose R is a total order on A and $B \subset A$. Then every element of B is either the smallest element of B or the largest element of B. Proof: Suppose $b \in B$. Let x be an arbitrary element of B. Since R is a total order, either bRx or xRb. Case 1: bRx. Since x was arbitrary, we can conclude that $\forall x \in B(bRx)$, so b is the smallest element of R. Case 2: xRb. Since x was arbitrary, we can conclude that $\forall x \in B(bRx)$, so b is the largest element of R. Thus, b is either the smallest element of B or the largest element of B. Since b was arbitrary, every element of B is either its smallest element or its largest element (Velleman, 2006)
Example 19: Lack of subject knowledge Theorem: Correct/Incorrect (no answer given) Reason; I can't say for sure as I can't remember the total order concept. Proof: Correct/Incorrect (no answer given) Reason; Likewise, i have no clear idea.

In the 11th question (see Table 13), most of the pre-service mathematics teachers gave incorrect answers, and many of them struggled to answer. The participants are probably unaware of total order, as shown by Example 19. As a result of examining the questionnaire, many participants said they had forgotten what they had learned about total order, which is a first-year subject. It is clear, from this, that participants rely on their memory rather than thinking and analysing the statement given to them.

Table 14

Example of Responses for Question 12

Theorem 12: What's wrong with the following proof that if $A \subseteq N$ and $0 \in A$ then $A = N$? Proof: We will prove by induction that $\forall n \in N\ (n \in A)$. Base case: If $n = 0$, then $n \in A$ by assumption. Induction step: Let $n \in N$ be arbitrary, and suppose that $n \in A$. Since n was arbitrary, it follows that every natural number is an element of A, and therefore in particular $n + 1 \in A$. (Velleman, 2006)
Example 20: Knowledge gaps in proof methods Theorem: Correct/*Incorrect* Reason; '-1' is an element of A but not an element of N. Proof: Correct/*Incorrect* Reason; We cannot prove such a theorem by induction.

The majority of the answers given for question 12 were correct. It was the pre-service mathematics teachers' knowledge of sets that allowed them to identify the mistake in the proof and answer the question correctly. In spite of the fact that most participants could see that the proof was incorrect, many were unable to explain how the mistake was caused by induction, as in Example 20. Accordingly, it is assumed that the pre-service mathematics teachers lack proof method knowledge.

The 13th question (see Table 15) was answered correctly by most participants. There was, however, a significant number of respondents who were unable to provide a sufficient explanation for their responses. The reason for this may lie in the lack of knowledge they have about proof methods, as illustrated in Example 21.

Table 15

Example of Responses for Question 13

Theorem 13: For all $n \in N$, $1.3^0 + 3.3^1 + 5.3^2 + \cdots + (2n+1).3^n = n.3^{n+1}$ Proof: We use mathematical induction. Let n be an arbitrary natural number, and suppose that: $1.3^0 + 3.3^1 + 5.3^2 + \cdots + (2n+1).3^n = n.3^{n+1}$. Then, $1.3^0 + 3.3^1 + 5.3^2 + \cdots + (2n+1).3^n + (2n+3).3^{n+1} = n.3^{n+1} + (2n+3).3^{n+1} = (3n+3).3^{n+1} = (n+1).3^{n+2}$ (Velleman, 2006)
Example 21: Knowledge gaps in proof methods. Theorem: Correct/*Incorrect* Reason; It isn't correct for $n = 3$. Proof: *Correct*/Incorrect Reason; Induction conducted correctly.

4. Discussion

This study examined the ability of pre-service mathematics teachers to decide if given theorems and proofs are correct and to justify their decisions. Despite the fact that the participants could often identify that the theorems and/or proofs were wrong, their reasons were often inadequate. According to this research, pre-service mathematics teachers are capable of identifying incorrect theorems and proofs, but not of explaining why they are incorrect (Ko, 2010; Ko & Knuth, 2009). As a result of their research, Uygan et al. (2014) examined the proving processes of pre-service mathematics teachers and concluded that they believed their proof skills were inadequate and made mistakes when evaluating reasons disrupting the axiomatic structure. Teachers in the study are perceived to be inadequate in making mathematical proofs, similar to pre-service mathematics teachers in the current study who are unable to evaluate proofs accurately.

Pre-service mathematics teachers' struggle was mostly the result of a lack of knowledge of the subject or proof methods, which caused them to use counter examples, intuition, and visualization by means of shapes or schemas when evaluating proofs. Different classes did not score significantly differently in terms of correct answers, according to the research. As a result of their answers to some questions, it was evident that the pre-service mathematics teachers were unaware of how proofs work and that they mostly evaluated proofs based on results, which was likely due to their primary reliance on rote learning. Due to their focus on the result, rather than examining the inductive and axiomatic structure of the proof, they overlook the incorrect parts. Unlike the current study, Doruk and Kaplan (2013) examined the proof evaluation skills of six pre-service mathematics teachers through semi-structured interviews. The results of the interviews conducted for proof indicated that the pre-service mathematics teachers were not able to evaluate the proof. This was attributed to the fact that their knowledge was not sufficient to generate key ideas, so they memorized the proofs instead. Pre-service mathematics teachers also prefer memorizing theorems without understanding the logic of the proof, as shown in the study. According to Doruk and Kaplan (2013), teachers should emphasize proof more in their classes and use more proof in the evaluation process of lessons to develop positive views of proof. Unlike the present study, Selden and Selden (2003) analyzed a single theorem in detail based on four arguments produced by eight mathematics teachers. According to the study, teachers tend to focus on superficial features of arguments and their ability to determine whether those arguments are proof is limited. It is similar to what was found in the study. An interview form was used by Miral (2013) to analyze the views of mathematics teaching students about mathematical proof methods. In interviews with 10 students, Miral (2013) reported that the students were positive about proof methods, but they had issues applying them effectively. The results of the current study are also consistent with this finding. Despite using incorrect theorems and proofs to examine the proof evaluation skills of pre-service mathematics teachers in the current study, it is generally concluded that teachers are lacking in proof evaluation skills when using various data collection tools and methods.

Question 5 and Question 9 were mostly answered correctly by pre-service mathematics teachers. Shapes and schemas made it easier for participants to reach correct conclusions since these questions were suitable for visualizing. According to Almeida (2000), visual arguments can have additional benefits by reinforcing conceptual imagery and reducing the difficulties students have with proof.

Among the difficulties students face while making mathematical proofs, Moore (1994) identified the following: not being able to express definitions, not understanding the meaning of concepts intuitively, not being able to use concept images while proving, a lack of generalization and using examples, not being able to use proof structures from definitions, not understanding mathematical language and notations, and not knowing how to start a proof. Additionally, due to the abstract nature of mathematics courses at universities, students may not understand the nature of proofs and may try to prove mathematical rules by memorization without utilizing proof techniques and strategies (Gibson 1998; Weber 2006). They also lack conceptual knowledge as to how and when to use the conceptual knowledge needed during the proof process, which is another reason why students have difficulty proving a theorem.

Teachers' performance on proofs increases when they evaluate proofs (Powers et al., 2010). Accordingly, this study will provide insight into the methods used in proof evaluation and the difficulties that arise, as well as eliminate deficiencies in this subject. Similarly, Almeida (2003) concluded that teachers' perceptions and experiences of proof are effective in the process of students acquiring skills while proving. Further, pre-service mathematics teachers' understanding of proof may be enhanced by examining and integrating such studies into lessons on proof in mathematics.

According to their answers to the questions, pre-service mathematics teachers lack the knowledge on the topic to provide effective evidence and support their arguments. However, there is no way to give a consistent result in this regard due to the fact that their answers are different from one question to another. To put it another way, pre-service mathematics teachers' knowledge of how to verify a claim changes independent of both the subject and method. In his study, Knapp (2005) identified two main reasons why students have difficulty proving their arguments. The first is that students lack knowledge of proof language and logic, and the second is that they lack the knowledge they need to construct definitions, theorems, and examples.

Study findings are limited to 13 proofs and theorems tested through the open-ended questionnaire. Therefore, tests created for different proof making methods should be used to examine pre-service mathematics teachers' proof evaluation competencies. Furthermore, semi-structured interviews would be useful for gathering more detailed information. Studies have found that mathematics teachers and pre-service mathematics teachers lack proof knowledge and have difficulty making proofs based on the results obtained in the examined studies. Nevertheless, there is little information about the methods used to verify these proofs. It is possible to gain more insight into this topic by conducting further research.

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