

## Research Article

# How do pre-service mathematics teachers respond to students' unexpected questions related to the second derivative?

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The aim of this study was to determine the how pre-service mathematics teachers respond to unexpected questions from students about the second derivative. A qualitative research method was used for this purpose, applying a case study framed as an in-depth analysis of how pre-service mathematics teachers respond to students' ideas. The participants were 39 pre-service mathematics teachers who were in their final year of their mathematics teacher education program. The pre-service teachers, who participated voluntarily, were asked to respond to open-ended questions relating to a scenario that included a teacher, as well as her 12th-grade students. The written responses that the participants provided constitute the data of this study. The results revealed that most of the participants could not effectively answer an unexpected question from students. Nearly half of the participants stated that they could not answer the question. Others ignored it, while some acknowledged the question and attempted to give an answer. Moreover, a small number of the participants made an effort to explain and demonstrate the concept of concavity by drawing the graphs of the function and relating them to the first derivative.

Keywords: Responding to student ideas; Unexpected question; Contingency; Knowledge Quartet; Pre-service mathematics teachers; Derivative concept

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## 1. Introduction

Mathematics teachers' knowledge and its role in enhancing students' mathematical thinking and learning plays an important role in mathematics education; as Even (1993) points out, teachers' professional knowledge is the most critical influence on students' learning. In consideration of this importance, Rowland, Huckstep, and Thwaites (2005) developed the Knowledge Quartet (KQ) to provide pre-service teachers with content-specific knowledge during the course of mathematics education. The KQ proposes a way of thinking about mathematics teaching in normal classroom settings, with a focus on the mathematics content of a lesson. This theory emphasizes the observation, analysis and development of mathematics teaching, with a focus on teachers' subject matter knowledge and pedagogical content knowledge, and it has therefore been regarded as an important theory in mathematics education (Breen, Meehan, O'Shea, & Rowland, 2018).

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More specifically, the KQ provides a means of organizing the situations in which mathematics teachers' knowledge "plays out" in the practice of teaching (Rowland, & Zazkis, 2013) and consists of four broad units: Foundation, Transformation, Connection, and Contingency. Foundation, the first unit, involves a theoretical background about mathematical content, as well as beliefs regarding mathematics and mathematics teaching. The remaining three units originate from this underpinning. Transformation, in this regard, comprises the ways in which knowledge can be transmitted from teachers to learners; the use of examples and selection of procedures to form concepts; and the choice of illustrations and representations. Connection, on the other hand, involves decisions about sequencing subjects or lessons; associating lessons with previous lessons and with students' existing knowledge; associating procedures with concepts; and anticipating and carefully sequencing the introduction of complex ideas in a lesson. Furthermore, Contingency, which constitutes the final unit of the Knowledge Quartet and which is the focus of this study, concerns unplanned examples in lessons, such as students' unexpected ideas and deviations from a lesson agenda in response to an unplanned opportunity (Rowland, Huckstep, & Thwaites, 2003; Turner, 2007). With respect to Contingency, MacDonald (1993) reports that many undergraduate students studying in teacher education programs feel unprepared to address the problems they may face in their professional lives. In this sense, teaching requires the ability to attend to students' questions, anticipate the difficulties that may occur, and deal with unpredictable, contingent events in the classroom. In the process of developing these skills in pre-service teachers, it is important to determine and evaluate how they respond to contingent actions, as well as to share prospective outcomes arising from these evaluations.

Evaluating pre-service teachers with respect to the Contingency unit of the KQ may be carried out according to four codes, defined as (a) *responding to students' ideas*, (b) *deviating from the agenda*, (c) *responding to the (un)availability of tools and resources*, and (d) *teacher insight* (<http://www.knowledgequartet.org>). The code of *responding to students' ideas* relates to the ability of teachers to give convincing, reasoned, and informative responses to the unexpected ideas and suggestions of students (Rowland, Thwaites, & Jared, 2011). The code of *deviation from the agenda*, moreover, focuses on how to overcome a situation that necessitates diverging from the designated lesson agenda (Kula, 2011). On the other hand, the code of *responding to the (un)availability of tools and resources* involves the tools and resources teachers use to make abstract concepts concrete (Rowland, Thwaites, & Jared, 2015). Such tools, resources and materials may consist of the main tools designated for the lesson agenda, or they may be drawn into lessons in an opportunistic way (Rowland, Thwaites, & Jared, 2011). The final code, *teacher insight*, denotes teachers' recognition of deficiencies in the lesson agenda they have prepared during the course of lesson (Rowland et al., 2009). Although expert teachers might easily recognize such shortcomings in their teaching, this awareness is less commonly observed in novice teachers (Rowland, Thwaites, & Jared, 2015).

The current study focuses on the first code relating to the Contingency unit of the KQ, *responding to students' ideas*, or teachers' responses to the thoughts produced by students in order to provide their mathematical development (Kula & Bukova Güzel, 2014). Turner (2009) points out that responding to students in this manner ensures more meaningful teaching; moreover, passing over such opportunities, or simply ignoring them or dismissing them as wrong, can be interpreted as a lack of interest (Rowland et al., 2009). Moreover, according to Thwaites, Huckstep, and Rowland (2005), when students articulate unexpected ideas, teachers acquire additional information about the nature of their existing knowledge. In this respect, it is believed that facing contingent situations in the teacher training process may provide pre-service mathematics teachers with opportunities to improve their pedagogical knowledge (Rowland, Thwaites, & Jared, 2015).

While Brown and Wragg (1993) suggest that responding to students' ideas is crucial to the outcome of a lesson, this skill is difficult for novice teachers to attain (Rowland, 2013). To address this issue, this study attempted to examine how pre-service teachers respond to unexpected student questions as a means to explore how they may address such unanticipated situations in their future teaching. In doing so, the author chose to provide an established scenario to the pre-

service teachers who participated in the study. While observations of lectures could provide more authentic information in this sense, it would be necessary to wait for an unexpected situation in the course of an observed lesson. Additionally, it would have been overly time-consuming to observe a large number of pre-service teachers in the classroom. As such, a pre-determined scenario was viewed as a more efficient means for observing pre-service teachers' responses to unexpected questions from students.

This study is focused on the derivative, one of the basic concepts of calculus that analyzes how things change and the rate at which they change (Orton, 1983; Tall, 1996). According to the literature, students have various difficulties understanding the derivative concept; these problems have been linked to the following issues:

- the modes of representation,
- the relationship to the slope of a tangent,
- the relationship to the concept of rate of change,
- the relationship to the derivative of a function at a point,
- the global view of the derived function, and
- the relationship between the first and second derivative of a function (Artigue, 1991; Aspinwall & Miller, 2001; Orton, 1983; Sánchez-Matamoros, García & Llinares, 2006, 2008).

It has also been reported that students encounter problems connecting the relationship between a function and a derived function to the relationship between the first and second derivative of a function (Baker, Cooley, & Trigueros, 2000; García, Llinares, & Sánchez-Matamoros, 2011). Therefore, in this study, the author aimed to determine how pre-service mathematics teachers would respond to an unexpected question from students. To this end, she designed a question that students might ask their teachers in order to investigate their responses. Accordingly, the research question explored in this study is "How will pre-service mathematics teachers explain to their students the relationship between the intervals where the graph of a function is concave up and concave down and the second derivative of the function?"

### 1.1. Responding to Students' Ideas

Teachers' typical responses to unexpected ideas and suggestions from students are (a) to ignore, (b) to acknowledge but put aside, and (c) to acknowledge and incorporate (Rowland, Thwaites, & Jared, 2015). Rowland and Zazkis (2013) state that there are sometimes good reasons for choosing "to acknowledge but put aside," and perhaps even "to ignore," but both mathematically and pedagogically, "to acknowledge and incorporate" is a more interesting approach to exploring possibilities. In this regard, Rowland, Thwaites, and Jared (2011) found that the great majority of the contingent moments leading to such responses are triggered by unexpected contributions from students. These triggers may be (a) students' responses to a question from the teacher, (b) students' spontaneous responses to an activity or discussion, and (c) students' incorrect answers to a question or as a contribution to a discussion.

Pre-service mathematics teachers' responses to students' ideas while teaching the limit concept have been subcategorized by Kula (2011) and Kula Ünver and Bukova Güzel (2016) as (a) repeating students' ideas, (b) approving students' ideas, (c) explaining and expanding on students' ideas, (d) answering students' questions, (e) asking how students came about their ideas, (f) correcting mistakes in students' ideas, and (g) ignoring students' ideas. Kula (2014), moreover, conceptualize the approaches of pre-service mathematics teachers towards the contingencies encountered during the teaching process within the context of responding to students' ideas, proposing the following sub-codes: (a) approving, (b) repeating, (c) giving clues, (d) reminding, (e) explaining and expanding, (f) using questioning, (g) discussing with the class, (h) adding examples, (i) using different representations, (j) considering different ideas and solutions, (k) answering questions, and (l) ignoring. In the same study, Kula also described the triggers that cause the pre-service teachers to exhibit these approaches in detail.

Thus far, the sub-codes relating to pre-service teachers' responses to students' ideas have been given. In the following section, the author would like to discuss some of the research about pre-service teachers' specific responses to students' ideas. Petrou (2013a), for example, examined a pre-service teacher who was teaching problem-solving using Schema-Based Instruction with 5th-grade students (ages 10-11). In the lesson, the students were given a diagram and asked to develop their own problems. Only one of the problems suggested by the students was correct; however, the pre-service teacher accepted all of them as correct without exploring them further. Moreover, the suggested problems were not discussed in the class. The researcher asserted that the participant's unwillingness to discuss the problems could be attributed to her own limited understanding. In addition, the fact that she did not discuss the students' problems prevented the other students from noticing the mistake.

A similar situation was observed in Kleve's (2013) study, which examined a prospective teacher's (Hans) lesson taught in the 5th grade (ages 11-12) about calculating with fractions greater than one. A mismatch between Hans' intention for the task and his students' answers for this task occurred. Expressing that he was very curious about the reasons for a student's answer, Hans aimed to reveal the underlying thought process. However, he never acknowledged the student's conception of the task. In that respect, the appropriateness of Hans' response may be questioned. Therefore, while it was important that Hans dealt with his student's answer, his failure to open a discussion about the reason for the answer was attributed to possible deficiencies in the Foundation dimension of the Knowledge Quartet.

In another study, by Corcoran (2013), the researcher focused on the responses of Róisín, who was a second year Bachelor of Education student teacher who was instructing teaching 3rd-grade students (8-9 years old) about the equivalence of fractions. In her lesson, Róisín expected her students to move too quickly through important mathematical ideas, and when faced with an unexpected situation, she asked the students to repeat their answers. Róisín adhered to her teaching agenda, did not deviate from her lesson plan, and in some instances, provided no explanations for the students. Moreover, she did not spend time assessing the students' understanding of the equivalence of fractions. Due to the absence of concrete explanations, the students had difficulty understanding the lesson.

Rowland, Jared, and Thwaites (2011), moreover, observed the lesson of a pre-service secondary mathematics teacher named Heidi. Although Heidi's prior expectations restricted her ability to address certain cases, her openness to students' suggestions and ability to anticipate where they would lead was very characteristic of Heidi's approach. In her teaching, she acknowledged unexpected suggestions, but she then put them aside.

On the other hand, a study by Rowland, Thwaites, and Jared (2015) disclosed situations in which a pre-service teacher both acknowledged and incorporated the unexpected responses into the lesson. In this case, a primary-school teacher trainee named Chantal acknowledged a student's idea; and although she had not included this issue in her planning, she deviated from the lesson agenda to respond to the student's unexpected answer and incorporated it in the lesson.

In another study, Petrou (2013b) examined a pre-service teacher's teaching approach with a 6th grade class (ages 11-12). The participant gave the students a table with goal scores and asked them to calculate the average of seven games. One of her students stated that the total number of goals should be divided by 7, and the other that it must be multiplied by 7. In the face of this unexpected situation, the participant asked both of her students to try both ways. When multiplying by 7, the average value was greater than the number of goals scored. This approach of the participant helped the students realize why multiplication was not a reasonable answer. As such, it was concluded that the participant supported the students by explaining the reasons for their answers; this and contributed to their ability to deal with opposing views.

## 2. Method

The case study research methodology may be used to determine how pre-service teachers respond to students' unexpected questions about the second derivative. Therefore, for the current research project, a case study approach was conducted in the form of an in-depth analysis of how pre-service mathematics teachers responded to students' ideas.

### 2.1. Participants

The participants in this study were 39 pre-service mathematics teachers in their final year of a high school mathematics teacher education program who volunteered to take part in the study. The participants had taken mathematics courses such as Calculus, Analytical Geometry, Discrete Mathematics, Differential Equations, Linear Algebra, Algebra, and Complex Analysis. In addition to the mathematical content, they took courses that aimed to prepare them for mathematics teaching, such as Mathematical Modeling, Mathematical Problem Solving, Mathematics and Art, Mathematics and Games, History of Mathematics, New Approaches in Mathematics, Introduction to Educational Sciences, Curriculum Development, Assessment and Evaluation, Classroom Management, Guidance, Teaching Methods in Mathematics, and Examination of Mathematics Textbooks. Apart from these courses, in their final year of study, the pre-service teachers were taking courses involving school-based placement titled School Experience and Teaching Practice. The study was carried out within the context of a course called Teaching Calculus course in the final week of the semester. This course is intended to prepare students to effectively teach calculus-based topics such as functions, limits, derivatives and integrals. In order to preserve their anonymity, the pre-service teachers (PT) names were kept confidential; in the reporting, they are designated as PT1, PT2, and so on.

### 2.2. Data Collection Tools

In Turkey, students attend high school for a period of four years, from age 15 to 18 (grades 9-12). For the purposes of this study, the pre-service teachers were provided with a scenario that involved a 12<sup>th</sup>-grade teacher and her students. In accordance with the National High School Mathematics Curriculum of Turkey, the derivative concept is taught in the 12<sup>th</sup> grade, which is the final year of high school. The scenario was prepared in the context of the following learning objective: "The student explains the concept of curvature on the graph of a function and determines the intervals that are convex up and convex down according to the sign of the second derivative" (see Appendix 1). This scenario, as well as its accompanying open-ended questions, was designed to reveal the pre-service teachers' views on the questions. Prior to the study, a mathematics teacher educator who has worked on KQ was asked to evaluate the appropriateness of the open-ended questions. It was agreed that the scenario and related questions were suitable to reveal the views of the participants with respect to responding to students' ideas.

In the scenario, a student asked the teacher how to make a proof of the second-derivative test for concavity. The participants were asked to write in detail how to respond to the student by putting themselves in the position of the teacher. In addition, they were asked to comment on whether they would plan their lesson differently for the same objective in light of the student's question. The written responses from the pre-service teachers constitute the data for this study. Ordinarily, qualitative research methodologies require data to be triangulated. However, this study is limited in the sense that the qualitative data were acquired from a single instrument. This was due to the fact that, because each pre-service teacher could not be expected to encounter the same unexpected situation in a classroom environment, the data were obtained from a single scenario, rather than real-life situations.

### 2.3. Data Analysis

In the course of analyzing the data, the researcher read the responses of the participants multiple times to become familiar with them. Through this process, certain patterns of responses emerged.

Categories describing these response types were then determined via content analysis. The main categories generated from the responses were compared with each other at every step, and the analysis process continued in this fashion. In traditional scientific research, the use of multiple coders and calculation of inter-coder consistency is emphasized in traditional research in order to minimize investigator bias (Patton, 2002). Accordingly, a mathematics teacher educator was asked to encode 10 randomly selected answer sheets. The inter-coder reliability formula, stated as “reliability = number of agreements / total number of agreements + disagreements,” was used for calculating the reliability of the two encoders (Miles & Huberman, 1994). The findings revealed an inter-coder reliability of .81. A consensus was then reached on the differences regarding the categories, and thus, the ways that the participants responded to the unexpected student question about the second derivative were identified. Since repeated analysis provides a more saturated and deep interpretation of the data (Miles & Huberman, 1994), the researcher analyzed all the data once again and rechecked the encodings.

The categories that emerged from the data analysis are presented in Table 1 and Table 2, illustrating the ways that the pre-service teachers responded and the frequency of the response types. In presenting the results, these categories have been exemplified with direct quotes from the answer sheets.

### 3. Results

The analysis of the answers given by the prospective mathematics teachers regarding the proof is provided in Table 1. When the responses of the participants were examined, it was determined that they did not have clear information about the proof of the second derivative. While some of the participants tried different ways to prove it, others stated that they could not prove it.

Table 1

*The Ways Pre-service Mathematics Teachers Address a Student's Unexpected Question*

Codes	Pre-service mathematics teachers	f
Writing that they did not know	4-5-6-8-9-11-14-16-17-20-21-22-24-26-27-28-29-35-36-37	20
Repeating the rule	2-7-9-15-19-23-24-26-29-30-31-33-36-38	14
Giving as homework	4-11-12-18-20-21-22-24-25-27-34-35-39	13
Exemplifying	1-7-14-17-23-27-31-32-34	9
Making connections with the first derivative	10-13	2
Saying to students that they did not know	20-21	2
Saying that the answer is too long	3	1
Demonstrating with software	16	1
Asking the class to answer the question	28	1

Most of the participants could not effectively answer the student's unexpected question. In this respect, twenty of the participants wrote directly that they could not answer the student's question. For example, PT<sub>37</sub> stated that she could not provide a proof, as shown below. On the other hand, according to PT<sub>8</sub>, he could not explain the answer to this question; and he also stated that he did not have sufficient subject knowledge. However, he added that he hoped to reach a sufficient level thanks to the Teaching Calculus course.

I could not prove it. (PT<sub>37</sub>)

Actually, I cannot explain. It is really sad that I am insufficient in subject knowledge, but I hope that I will increase my competence in this subject through the Teaching Calculus course. (PT<sub>8</sub>)

Seven of the 20 participants who said that they could not provide a proof stated that they would give it to the students as homework. For example, PT<sub>11</sub> stated that he could not carry out the proof at that time. He further explained that if he had experienced this unexpected situation in a real classroom environment, he would do the proof if he knew how, but if he did not know how to do it at that time, he would give it to the students as a homework assignment. In his words,

I don't know the proof right now. But if I encountered this question during the lesson, if I knew how, I would answer it immediately. But if I didn't know, I would give it to the students as homework. (PT<sub>11</sub>)

Moreover, five of the 13 pre-service teachers who stated that they would give their students the proof as homework gave no information about whether they could do the proof or not. One of these participants (PT<sub>27</sub>) tried to explain the relationship between the second derivative of the function and the intervals in which the function graph is concave up and concave down through the examples of the functions he drew. However, the others made no attempt to demonstrate this relationship. Furthermore, PT<sub>4</sub> reported that she would tell her students to research the proof, as the teacher's role in the constructivist approach is to be a guide:

In the constructivist approach, the role of the teacher is to assist students in assimilating new information, not to give information to them. Therefore, I [would] ask the students to research the answer to this question and share it with their classmates in the next lesson. (PT<sub>4</sub>)

In addition to these responses, fourteen of the pre-service teachers tried to address the student question by repeating the second derivative given in the scenario. However, when the answer sheets of these pre-service teachers were examined, it was determined that they could not perform the proof. For example, PT<sub>29</sub> verbally repeated the second derivative given in the lesson plan as follows, but he did not answer his student's question:

The second derivative of a function is the derivative of the derivative of that function. We assume a function  $f(x)$  is twice differentiable.  $f(x)$  is concave up over an interval if its derivative is increasing over that interval, in which case its second derivative is positive.  $f(x)$  is concave down over an interval if its derivative is decreasing over that interval, in which case its second derivative is negative. (PT<sub>29</sub>)

Participants PT<sub>20</sub> and PT<sub>21</sub> also stated that they would simply tell their students that they could not perform the proof. In this sense, PT<sub>21</sub> stated that he did not know the answer to the question, and that he would research it before the next lesson and discuss it with his students.

If I come across a question that I do not know or have not prepared before the lesson, as in the example, I would say clearly that I had not thought about it before. I would like to research this proof, and students should also research this question as homework. I will say we will discuss it in the next lesson. (PT<sub>21</sub>)

Likewise expressing that she could not answer the unexpected question of the student, PT<sub>24</sub> stated that she would consider the question, but she would not answer it in that lesson. She also indicated that she would tell her students she did not want to interrupt the lesson, and that she would answer this question at the end of the class. However, she also asserted that would not actually answer at the end of the lesson, and instead, she would give the proof during the following lesson. Although she did not express this directly, this appeared to constitute an attempt to gain time to research how to do the proof. In her words:

I cannot answer this question. That's why I would say to the students, "let's continue the lesson for now, I will answer in more detail at the end of the subject." But I won't finish the subject in that lesson. I will answer the question in the next lesson. (PT<sub>24</sub>)



Another of the pre-service teachers, PT<sub>28</sub>, stated that she could not answer the student's question at that moment, but that when she becomes a teacher, she will be prepared for the lesson beforehand. She further asserted that she would first ask her students how to carry out the proof, which was not included in the lesson plan. She then explained that she would wait for them to produce ideas and develop the proof, and then she would do it herself. As she put it:

I don't know the proof right now. But when I become a teacher, I will research this proof beforehand and prepare for the lesson. If I face such a question that is not included in my lesson plan, I will try to answer so that the students will understand the subject better. Before I answer the student's question myself, I will ask the class to generate ideas and to produce the proof. After the class discussion, I will prove it on the board. (PT<sub>28</sub>)

On the other hand, nine of the pre-service teachers took the first and second derivatives of the example function and tried to draw their graphs, attempting to answer the student question by looking at the derivatives and graphs of one or more functions. For instance, PT<sub>23</sub> answered the question to some extent by expressing the function  $f(x) = x^2$ . She took the first and second derivative of the function and tested the intervals on table, graphing the function  $f'(x) = 2x$  and showing that it is an increasing function. In this manner, she tried to relate the first derivative to the second derivative (see Figure 1).

Example: Let's discuss the concavity of function  $f$ .

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$f''(x) = 2.$$

$x$		0	
$f'(x)$	$\Rightarrow$		$\rightarrow$
$f''(x)$	$+$		$+$

Let's look at the graph of the function  $f'(x) = 2x$ . It's an increasing function.

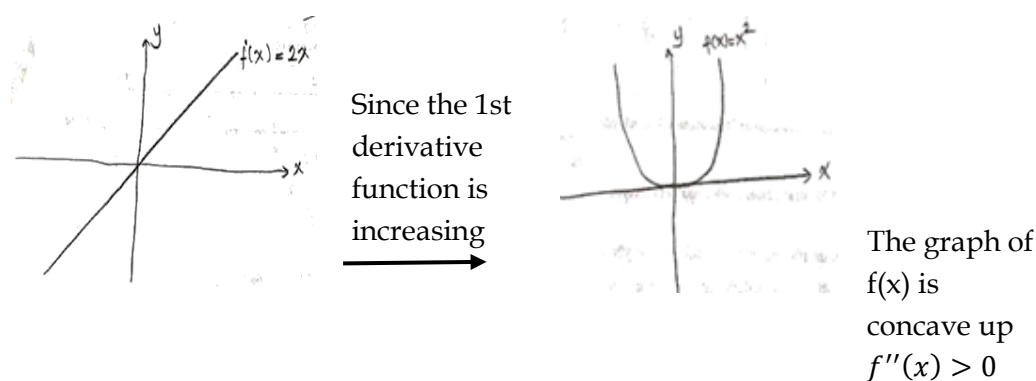
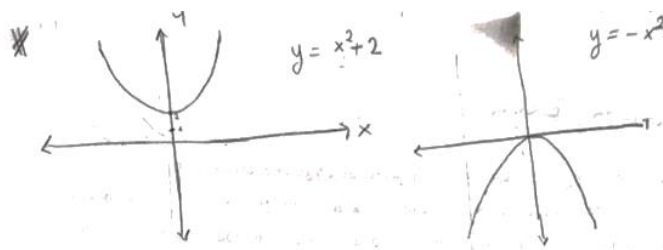


Figure 1. The response of PT<sub>23</sub>

Similarly, PT<sub>14</sub> drew the graphs of two functions and stated that he would ask his students to find the first and second derivatives of these functions. He asked them to decide whether the second derivative was greater than zero and determine whether the first derivative was increasing or decreasing by looking at the graphs. However, he did not explain how this questioning process would continue (see Figure 2).

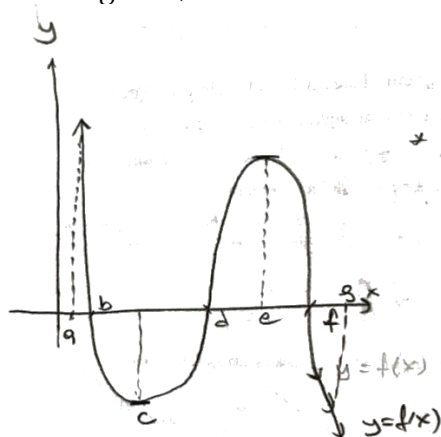




I would ask the students to draw the graphs of these two functions, then ask them to find the first and second derivatives and examine the intervals where the second derivative is negative or positive. In drawing the graph of the first derivative, I would want them to think about how the first derivative affects the increase and decrease.

Figure 2. The response of PT<sub>14</sub>

Only two of the pre-service teachers were able to show the relationship between the intervals where the graph of a function is concave up and concave down and the second derivative of the function. One of them, PT<sub>10</sub>, first determined the intervals at which the function increased and decreased by using the graph of the function she drew. On the graph of the first derivative function in Figure 3, she showed the concavity in increasing and decreasing intervals.



- Since  $f$  is a decreasing function at the interval  $(a, c)$ , then  $f'(x) < 0$  at the interval  $(a, c)$
- Since  $f$  is an increasing function at the interval  $(c, e)$ , then  $f'(x) > 0$  at the interval  $(c, e)$
- Since  $f$  is a decreasing function at the interval  $(e, g)$ , then  $f'(x) < 0$  at the interval  $(e, g)$

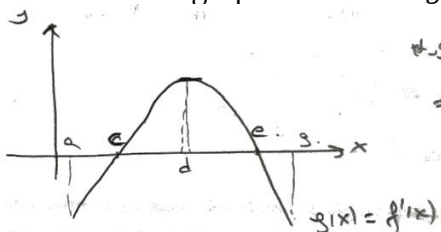
Let's show this in the table.

$f'(x)$	$-$	$+$	$-$
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$$f'(c) = 0$$

$$f'(e) = 0.$$

Let's draw the graph of function  $g(x) = f'(x)$ .

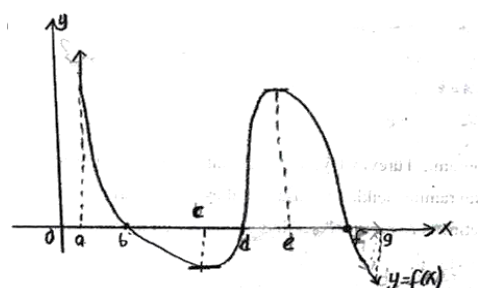


- Since  $g$  is an increasing function at the interval  $(a, d)$ , then  $g'(x) > 0$  At the interval  $(a, d)$ ;  
So  $f''(x) > 0$ .
- Since  $g$  is a decreasing function at the interval  $(d, g)$ , then  $g'(x) < 0$  at the interval  $(d, g)$ ;  
So  $f''(x) < 0$ .

Looking at the graph of the function  $f$ , you can see the intervals where it is concave up and concave down.

Figure 3. The response of PT<sub>10</sub>

Similarly, PT<sub>13</sub> tried to answer the student question using a graphical representation. As seen in the figure below, he drew a general function graph, and then he made geometrical interpretations by drawing the graph of the first derivative. Thus, he answered the student's unexpected question by using the relationships between the graphs as presented in Figure 4.



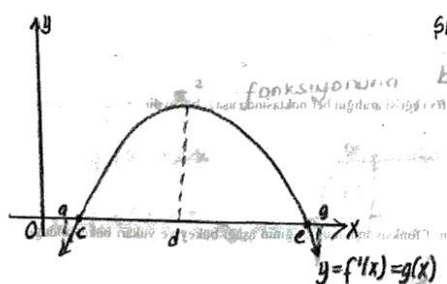
$f(x)$  is:

a decreasing function at interval  $(a, c)$ ,  
 $f'(x) < 0$

an increasing function at interval  
 $(c, e)$ ,  $f'(x) > 0$

a decreasing function at interval  
 $(e, g)$ ,  $f'(x) < 0$

Let's draw the graph of  $f'(x)$ .



$f''(x) < 0$  when  $g'(x) < 0$

$f''(x) > 0$  when  $g'(x) > 0$

Figure 4. The response of PT<sub>13</sub>

Another pre-service teacher (PT<sub>16</sub>) responded that he would draw a graph of the function using computer software, then determine the inflection points and intervals of the function on the graph and show the second derivative. He also explained that when the second derivative is positive, it indicates that the function is concave up, and when negative, the function is concave down. As he put it,

I would give an example by turning on the computer and using mathematics software. With the software, I would find the second derivative of the function after determining the inflection points and intervals. We would see that it is positive when concave up and negative when it is concave down. Another way would be to draw the graph of the second derivative function using the software. We would come to the same conclusion. (PT<sub>16</sub>)

Taking a different approach, PT<sub>3</sub> answered that he would tell his students that the proof was too long to explain; he also expressed no intention to perform the proof in later lessons. Because he let the student's question go completely unanswered, with no plan to address it at a later time, it can be inferred that he did not care about the question. In his words:

I'd say it's a very long proof. I would say that it will take a lot of time if I explain. (PT<sub>3</sub>)

After providing their responses to the unexpected question from a student in the given scenario, the pre-service mathematics teachers were asked whether they would prepare a different lesson plan than the one illustrated in the scenario. Six of the participants did not answer this question, while seven stated that they would make no changes to the lesson plan. However, another 26 of the participants reported they would develop a different plan. On the other hand, only 20 of them expanded on how this planning would be done. The views of these 20 pre-service teachers are included in Table 2.

Table 2  
*The Views of the Pre-Service Mathematics Teachers on Reorganizing the Lesson Plan*

Codes	Pre-service mathematics teachers	f
Being prepared to answer the question	2-4-5-6-7-10-13-14-16-26-28-29-30-33	14
Using technology assisted instruction	11-19-21-23	4
Using different function examples	27-34	2

As the table indicates, six of the pre-service teachers, PT<sub>9</sub>, PT<sub>12</sub>, PT<sub>15</sub>, PT<sub>32</sub>, PT<sub>37</sub> and PT<sub>38</sub>, left the question blank; moreover, none of these participants had successfully provided the proof. Another 7 of the participants stated that they would not plan differently for these objectives. None of these participants were able to provide the proof requested in the scenario, either. Five of the participants who expressed that they would not make changes in the lesson plan did state that they would attend the lesson well-prepared, considering the possibility of such a question. For example, PT<sub>22</sub> said that by researching from a few books, he would be prepared for questions that might come. As he explained:

I wouldn't have prepared a different lesson plan, but I would do research from a few books before the lesson in case of possible questions. At least, I would take a peek.  
 (PT<sub>22</sub>)

Of the 26 pre-service teachers who responded that they would prepare a different lesson plan, a total of 14 indicated that they would include this proof in their lessons. For instance, rather than saying that he did not know the answer, PT<sub>26</sub> clarified that it would be appropriate for the students to carry out the proof if they asked about it:

Obviously, when I am a teacher, I would have the proofs of the theorems with me without exception. Even if the proof is very difficult, even if I am sure that the students will not understand even if I explain it, I think it would be more appropriate to give the student the proof instead of answer that I don't know. Such things damage the student's trust in the teacher. (PT<sub>26</sub>)

PT<sub>30</sub>, on the other hand, stated that due to the possibility of encountering this question, she would give the proof first. In her words:

I would, in case I might encounter these questions frequently, I would provide the proof first. (PT<sub>30</sub>)

Another 4 of the pre-service teachers responded that they would organize their lesson plans to provide technology-assisted instruction. For example, PT<sub>23</sub> stated that he would show the relationship between the two objectives by using technology:

Instead of giving this objective directly, I would teach it by connecting it with the previous objective. In fact, since these concepts were not fully understood by the student, I could explain them by supporting them with technology-assisted instruction.  
 (PT<sub>23</sub>)

On the other hand, two of the pre-service teachers stated that they would plan the lesson to use different function examples. For example, PT<sub>27</sub> said that she would give different function examples starting from an easier level. As she put it:

I will give different function examples that the student can perform and understand more easily. I'll start from an easier level. (PT<sub>27</sub>)

## Discussion

In this study, the researcher examined the ways that pre-service mathematics teachers chose to answer an unexpected question asked by a student about the second derivative of a function. Almost half of the pre-service teachers stated that they could not answer this question; and while

some of the participants indicated that they would ignore the question altogether, others stated that they would acknowledge and try to answer it. In this respect, Kula (2014) identified the triggers that prompted pre-service teachers to answer students' questions as subjects that students do not understand or content that is not included in the lesson plan. The trigger in this study was the student question in the scenario.

A small number of the pre-service teachers tried to explain and demonstrate the concept of concavity by drawing the function graphics and relating them to the first derivative. In the study of Baker, Cooley, and Trigueros (2000), a total of 41 university students were unable to interpret the relationship of the second derivative to the first. In the literature, several similar difficulties were reported with regard to students' understanding of the connection between function's graph and its first derivative, and of the relationship between first and second derivatives (Aspinwall et al., 1997; Baker, Cooley, Trigueros, 2000; Garcia, Llinares & Sánchez-Matamoros 2011; Sánchez-Matamoros, Garcia & Llinares, 2006, 2008).

Likewise, in the current study, it was observed that the pre-service teachers did not express the relationship between derivative and slope. This may be because they did not realize the relationship, leading us to the possibility that they lacked sufficient subject matter knowledge. Although calculus instruction begins with using slopes of lines to develop derivatives, it seems that the pre-service teachers, although they had taken calculus, had problems understanding the concept of derivative (Stump, 1999). Accordingly, the weaknesses of the pre-service teachers' performance in terms of content knowledge also affected their pedagogical content knowledge (Güler & Çelik, 2018).

Some participants, on the other hand, took the derivatives of the example functions and drew the graphs of these functions, demonstrating whether the function was increasing or decreasing for the second derivative. However, it has been argued that showing the accuracy of a theorem through examples, or even a single example, may cause difficulties in students' perspective to generalization. In this sense, examples are valuable in terms of proof but not sufficient (Ozgur, Ellis, Vinsonhaler, Dogan, & Knuth, 2019). Mathematically, they may cause the misconception that it will be sufficient to prove or show a single example. However, a theorem or rule that is true for one example may not be true for a different example. As such, the realization of a proof by affordances of examples could lead to digression from the appropriate proof approach (Coe & Ruthven, 1994). From this point of view, it is important to identify and eliminate the deficiencies in pre-service teachers' perspectives of proof.

Given the results of the current study, it can be said that some of the participants were interested in student ideas, as with the pre-service teachers in the studies of Rowland, Thwaites, and Jared (2011) and Petrou (2013b). Stating that they would give the proof to their students as homework and then show the proof in the next lesson, the pre-service teachers expressed that this would give them time to be more prepared for the next lesson. In this regard, it can be asserted that these pre-service teachers acknowledged the student question, but they were unable to answer the question due to their lack of subject matter knowledge. In the results of Güler and Çelik's (2019) study, prospective teachers were successful at identifying student difficulties or misconceptions but their instructional explanations were found to be inadequate. Similar to Bukova Güzel's (2010) study, the pre-service teachers in this case displayed some knowledge of learners; however, this knowledge requires further development, implying that they may need additional training, particularly with respect to possible questions from students. Otherwise, these unexpected situations may adversely affect their teaching in the beginning years.

Some of the participants also tried to overcome the problem by repeating the rule, rather than answering the student question. In these instances, the pre-service teachers took the question into account but could not answer it. Similarly, in the studies of Petrou (2013a), Kleve (2013), Corcoran (2013), and Rowland, Jared, and Thwaites (2011), the pre-service teachers acknowledged the students' questions, but could not take their response further.

One pre-service teacher in the current study stated that showing the proof to his students using mathematical software would lead to greater understanding, while another reported that she would pose the question to the class, addressing the ideas of her students by creating a discussion environment and then giving the proof. However, she did not provide the proof on the answer sheet. As Kula (2014) points out in her study, discussion with the class is one way that teachers may respond to a student question. However, in this instance, it can be inferred that the pre-service teacher did not know the answer, because she did not carry out the proof. For this reason, further research may be carried out to explore what pre-service mathematics teachers may do when they encounter this question in a real classroom environment.

Another of the pre-service teachers expressed in his response that the proof was too long, and it would take too much time to prove and explain it. As with the cases reported by Kula (2011, 2014), he was not interested in student ideas, so he ignored the question and moved forward with the lesson. Still others of the participants, rather than indicating that they would ignore the student question, stated simply that they could not provide the proof. This situation gives insight into how the pre-service teachers' lack of subject knowledge may be reflected in the classroom environment.

As with the results of this study, Rowland et al. (2009) also concluded that responding to students' ideas occurred in three ways: ignoring, acknowledging but putting aside, and acknowledging and incorporating. According to Rowland and Zazkis (2013), there may be important reasons for choosing the second approach, but the third attracts more attention both mathematically and pedagogically. In this sense, Turner (2009) bases the ability to give appropriate answers to students, to some extent, on teachers' subject matter knowledge and pedagogical content knowledge. From this point of view, it can be stated that inadequacies in the subject matter knowledge and pedagogical content knowledge of pre-service teachers may affect their interest in addressing students' ideas. In this respect, it is important to ensure that pre-service teachers develop both the subject matter knowledge and the pedagogical content knowledge necessary to help them address unplanned questions.

The ways in which pre-service mathematics teachers answered the student's unexpected question about the second derivative in the given scenario were examined through their written responses. However, although they were given time to reflect on their responses, the pre-service teachers were generally unable to carry out the proof. While some of them did attempt to answer the question, others chose to put the question aside and ignore it.

Due to the large number of pre-service teachers, I chose to carry out the study with a written scenario to explore how they would respond to an unexpected question in general terms. On the other hand, while conducting microteachings with pre-service teachers has some limitations in terms of eliciting and responding to student ideas, the practical benefits outweigh the limitations (Kula Ünver, Özgür, & Bukova Güzel, 2020). As such, microteachings may provide an opportunity for pre-service teachers to encounter potential student questions. On the other hand, lesson observations may be a more effective approach to determine how pre-service teachers respond to students' ideas. As such, the current study is limited in that it was carried out in the context of a written scenario, rather than in a real classroom environment. To overcome this limitation, it may be appropriate in future studies to explore pre-service mathematics teachers' responses to unexpected student questions in a real classroom environment.

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## Appendix 1.

Jasmine was teaching applications of derivative with a 12<sup>th</sup>-grade (ages 17-18) class. A part of her lesson plan is given below. The objective of the lesson according to the National High School Mathematics Curriculum was “The student explains the concept of concavity on the graph of a function and determines the intervals that are concave up and concave down, according to the sign of the second derivative.” Jasmine started her teaching of second derivative of a function as follows:

### Second derivative:

Suppose that the first derivative  $f'(x)$  of a function  $f(x)$  exists in a closed interval  $[a, b]$ , and the second derivative  $f''(x)$  exists in an open interval  $(a, b)$ . Then the following sufficient conditions for the concavity are valid:

- a) If  $f''(x)$  exists and is positive on an open interval, then the graph of  $y = f(x)$  is concave up on the interval.





b) If  $f''(x)$  exists and is negative on an open interval, then the graph of  $y = f(x)$  is concave down on the interval.



**Example:**

Find the intervals for which the function  $f: \mathbb{R} \rightarrow \mathbb{R}$ ,  $f(x) = -2x^3 - 3x^2 + 12x + 1$  is concave up and concave down.

The first and second derivatives of  $f(x) = -2x^3 - 3x^2 + 12x + 1$  are;

$$f'(x) = -6x^2 - 6x + 12 \quad \text{and} \quad f''(x) = -12x - 6$$

$$\text{If } f''(x) = 0 \Rightarrow x = -1/2 \text{ then } f(-1/2) = -11/2$$

The table of change for the second derivative of the function is as follows.

	$-\infty$		$-1/2$		$+\infty$
$f''(x)$		+	0	-	
$f(x)$	$+\infty$	U	$-11/2$	n	$-\infty$

The graph of  $y = f(x)$  is concave up on  $(-\infty, -1/2)$ .



The graph of  $y = f(x)$  is concave down on  $(-1/2, +\infty)$ .



Jasmine conducted her lesson in accordance with her plan. After solving the example, Isaac, one of the students in the class, asked Jasmine, "How can we prove that the second derivative of a function is positive when its graph is concave up, and its second derivative is negative when it is concave down?"

**Questions:**

- 1) How would you answer Isaac's question that was not included in Jasmine's lesson plan? Please explain in detail.
- 2) Would you make a different plan in your pre-lesson preparation for the same outcome? Please explain in detail.