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Research Article

Algebraic thinking in the context of geometry and measurement: Eighth-grade students' reasoning processes

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The aim of this study is to examine how eighth-grade students use symbols and algebraic relationships, patterns and generalization, and multiple representations in geometry and measurement tasks within the framework of algebraic thinking, and to identify the conceptual difficulties that emerge during this process. In the study, algebraic thinking was addressed through the components of symbols and algebraic relationships, patterns and generalization, and multiple representations. The study was conducted using a basic qualitative research design. The data were collected through clinical interviews, video recordings, and student worksheets from six eighth-grade students attending a public school with a middle socio-economic profile. The data were analyzed using thematic analysis. The findings indicate that students' algebraic thinking levels differ markedly in geometry and measurement tasks in terms of their ways of using and interpreting formulas, simplifying algebraic expressions, constructing equations appropriate to verbal situations, interpreting the meanings of literal expressions, making generalizations for near and far terms in patterns, and transitioning between representations.

Keywords: Algebraic thinking; Eighth grade; Geometry; Measurement; Multiple representations

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1. Introduction

Algebraic thinking, considered a dimension of mathematical thinking, refers to a process of reasoning that extends beyond arithmetic, beginning with the generalization of operations and their properties, questioning the meaning of equality, engaging with unknown and varying quantities and operations, and extending to the structuring of functions and relationships. This process requires the development of ways of thinking such as analyzing relationships among quantities, examining change, generalizing, justifying, and validating (Blanton & Kaput, 2011; Kaput, 2008; Kieran, 2022). This multidimensional structure makes it difficult to address algebraic thinking under a single comprehensive definition (Kaput, 1999).

Recent theoretical studies have more clearly articulated the multidimensional nature of algebraic thinking. Kieran (2022) conceptualizes this form of thinking in terms of analytic, structural, and functional dimensions and emphasizes that generalization constitutes a central process across all these dimensions. Similarly, Kaput (2008) explains algebraic thinking through the components of generalized arithmetic, functional thinking, and modeling, while Blanton and Kaput (2011) focus on the structuring of relationships among quantities through representations.

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These perspectives indicate that algebraic thinking is not merely a process of performing symbolic operations; rather, it involves constructing relational meaning, developing a sense of structure, and engaging in abstraction through transitions among representations.

Algebraic thinking is regarded as a relational structure that connects different domains of mathematics and becomes particularly visible in the contexts of geometry and measurement (Kieran, 2022; Van de Walle et al., 2012). In these contexts, students are expected to establish relationships among quantities, develop generalizations, and interpret relationships among variables. Recent studies explain this cross-domain connection particularly through multiplicative thinking and demonstrate that when students develop proportional structural sense, they are better able to translate geometric situations into more robust algebraic models (Day et al., 2024).

However, studies that directly examine the manifestations of algebraic thinking in geometry and measurement contexts remain limited (Day et al., 2024; Kieran, 2022). Research on early algebra has largely focused on arithmetic-based generalization processes and functional relationships, while the use of algebraic expressions, generalization processes, and transitions among representations within geometry and measurement contexts have remained relatively underexplored. Moreover, although developmental studies (Sun et al., 2023) indicate that algebraic thinking progresses from arithmetic approaches toward symbolic generalization, the cognitive manifestations of this progression in geometry and measurement contexts have not been examined in sufficient detail.

In this regard, the eighth-grade level can be considered a critical transition stage in which students both engage systematically with algebraic expressions and make intensive use of algebraic relationships in geometry and measurement contexts (Kieran, 2004; Sun et al., 2023; Van de Walle et al., 2012). Therefore, a detailed investigation at this level is important for revealing the structural manifestations of algebraic thinking in geometry and measurement contexts. Accordingly, this study aims to examine how eighth-grade students with different levels of algebraic thinking activate and coordinate components of algebraic thinking in problem situations within the context of geometry and measurement.

2. Dimensions of Algebraic Thinking in Geometry and Measurement Contexts

This study is grounded in the conceptual intersections among existing theoretical frameworks and focuses on shared epistemological themes beyond terminological differences. In this context, algebraic thinking is conceptualized under three core dimensions through a synthesis of the relevant literature: (i) symbols and algebraic relationships, (ii) patterns and generalization, and (iii) the use of multiple representations.

These dimensions were identified through a joint consideration of Kaput’s (2008) emphasis on generalized arithmetic and functional relationships, Blanton and Kaput’s (2011) focus on representing relationships among quantities within functional thinking, and Kieran’s (2022) distinction among structural, analytic, and functional dimensions of algebraic thinking. In addition, the cross-domain continuity of quantitative structures emphasized by Day et al. (2024) supports the view that these three dimensions provide an analytic framework applicable to geometry and measurement contexts as well.

Recent theoretical studies indicate that differentiating algebraic thinking into such dimensions is not merely a categorical classification but also serves as a functional model for describing students’ cognitive development. Sun et al. (2023) propose a developmental model progressing along the axes of generalized arithmetic, functional relationships, and quantitative reasoning, demonstrating that symbolization and representational diversity vary systematically with students’ thinking levels. This developmental perspective suggests that the three-dimensional structure adopted in the present study is grounded not only theoretically but also developmentally.

2.1. Symbols and Algebraic Relationships

One of the fundamental components of algebraic thinking is the interpretation of the concept of variables and algebraic expressions. Depending on the mathematical context, a variable may assume different roles, such as an unknown, a generalized number, a placeholder, or a parameter (Kieran, 1981; Kuchemann, 1978). It is critical for students to view algebraic expressions not merely as symbolic strings on which procedures are performed, but as structures representing relationships among quantities, for the development of the structural dimension of algebra (Sfard & Linchevski, 1994; Tall & Thomas, 1991).

In the context of geometry and measurement, this issue becomes particularly salient both in interpreting the meaning of the variable and in writing and making sense of algebraic expressions, especially when establishing functional relationships among quantities. For example, calculating the perimeter of a triangle whose side lengths are represented algebraically, interpreting how the perimeter changes depending on the values assigned to the variable, or symbolically expressing the relationship between the number of sides and the number of diagonals in polygons requires students to conceptualize variables as quantities embedded in relationships (Kaput, 2008; Kieran, 2022).

2.2. Patterns and Generalization

Generalization is one of the central processes of algebraic thinking (Kaput, 1999). Patterns are regarded as one of the natural contexts for generalization (Van de Walle et al., 2012). In growing geometric patterns, students are expected to articulate the functional relationship between the step number and the associated quantity when formulating generalizations (Grandau & Stephens 2006; Türkmen & Tanışlı, 2019). Studies aimed at identifying and generalizing relationships among quantities within patterns provide important insights into mathematical relationships and functions.

Research (e.g., Rivera, 2010) emphasizes that growing geometric patterns, compared to numerical patterns, provide stronger support for the generalization of functional relationships between two quantities. By examining the structure of a figure within a geometric pattern and analyzing invariant and varying relationships within that structure, students are able to generalize numerical relationships based on their structural analysis.

Day et al. (2024) demonstrate that multiplicative thinking constitutes the cognitive foundation of generalization, particularly in situations involving geometric growth. It is reported that developing scaling-based and proportional structure sense enhances students' capacity to translate linear and multiplicative relationships into symbolic models. This finding indicates that patterns and generalization should be addressed not only within the context of linear growth but also within multiplicative structures.

Instructional designs based on geometric patterns similarly support students in recognizing structural regularities and expressing these structures symbolically (Apsari et al., 2020). These studies suggest that visual representations may play not only a supportive but also a constitutive role in the generalization process.

For example, in a growing pattern composed of squares, while the number of squares may increase linearly, the perimeter may not increase by a constant amount at each step. This situation requires students to distinguish between varying quantities, identify invariant relationships within the structure, and reason comparatively about area and perimeter.

As emphasized by Day et al. (2024), a given growing geometric pattern can be redesigned to embody a multiplicative structure. For instance, a configuration formed by a 1×2 rectangle in Step 1, a 2×3 rectangle in Step 2, and a 3×4 rectangle in Step 3 yields an area that can be expressed algebraically as $A(n) = n(n + 1)$. Such structures illustrate how multiplicative relationships support the process of generalization.

In this case, the student is required to reason multiplicatively about the relationship between two quantities (length and width). In this respect, patterns and generalization in the context of geometry and measurement contribute to the development of variable thinking by making functional and structural relationships among quantities more explicit.

2.3. The Use of Multiple Representations

One of the fundamental components of algebraic thinking is the ability to use, interpret, and transition among mathematical representations. Representations enable mathematical ideas to be expressed in different forms and serve as bridges between abstract structures and concrete situations (Kaput, 2000). According to Ntsohi (2013), algebraic thinking involves representing information mathematically through verbal expressions, diagrams, tables, graphs, and equations, and analyzing different situations through these representations (National Council of Teachers of Mathematics [NCTM], 2000; Panasuk, 2010). This process encompasses not only the use of symbols but also determining unknown values, establishing relationships, interpreting functional relationships, and engaging in mathematical justification.

The use of representations plays a central role in the development of algebraic generalizations. Students often first notice relationships among quantities verbally or visually and subsequently structure these relationships through tables, graphs, or symbolic expressions. In this respect, representation is not merely a means of expression but a cognitive environment in which relational thinking develops.

Sarmasági et al. (2025) identify representation transformation as a shared cognitive foundation of both algebraic and computational thinking and demonstrate that students' ability to transition systematically from one representation to another significantly influences their generalization performance. This perspective indicates that the use of multiple representations is not merely a pedagogical tool but a fundamental component of cognitive structuring.

The context of geometry and measurement provides a learning domain in which the use of multiple representations emerges naturally and in an integrated manner. For example, students may visualize a problem through a geometric figure (Blanton et al., 2015; Van de Walle et al., 2012) and algebraically justify how the area of a triangle can be calculated.

In such processes, transitions between visual and symbolic representations enable students to justify mathematical relationships more explicitly. For this reason, within the scope of this study, the use of multiple representations is conceptualized not merely as a subskill of algebraic thinking but as a structural component integrated with processes of symbolic meaning-making and generalization.

3. Current Study and Research Questions

Examining algebraic thinking in the context of geometry and measurement does more than establish connections between two domains of mathematics; it also makes the process of constructing algebraic structure more visible. Geometric and measurement contexts enable relationships among quantities to be traced through visual and structural representations, thereby making students' processes of generalizing, identifying variables, and establishing relationships more concrete. However, in the teaching of geometry and measurement content, the use of formulas is sometimes addressed primarily at a procedural level, which may limit students' ability to make sense of the structural relationships underlying symbolic expressions (Sfard & Linchevski, 1994; Tall & Thomas, 1991). Therefore, a detailed examination of algebraic thinking processes emerging in geometric contexts is important for understanding both students' generalization strategies and their ways of constructing symbolic relationships.

In this respect, the study aims to contribute to explaining how algebraic thinking manifests in geometry and measurement and to providing a more in-depth account of the cognitive processes involved in these learning domains. Within this context, the eighth-grade level constitutes a critical stage in which students engage systematically with algebraic expressions and the concept of a variable, while also making intensive use of algebraic relationships in geometry and measurement contexts (Kieran, 2004; Van de Walle et al., 2012;). Furthermore, developmental models indicate that students' algebraic thinking progresses from arithmetic-based approaches toward symbolic generalization, and that this differentiation becomes more pronounced at higher grade levels (Sun et al., 2023).

Although recent literature emphasizes the cross-domain nature of algebraic thinking, it also indicates that studies examining algebraic thinking processes in geometric contexts at the middle school level remain limited (Day et al., 2024; Kieran, 2022). This gap highlights the need to investigate how higher-level symbolization and structural sense processes identified in developmental models manifest within geometric contexts. In this regard, a detailed investigation at the eighth-grade level appears theoretically warranted. Accordingly, both theoretical and pedagogical considerations necessitated focusing the study on the eighth-grade level.

In line with this rationale, the central problem of the study is formulated as follows: How do eighth-grade students with different levels of algebraic thinking activate and coordinate components of algebraic thinking in problem situations within the context of geometry and measurement? Based on this central problem, the study addresses the following questions:

RQ 1) What are the algebraic thinking levels of eighth-grade students?

RQ 2) How do students with different levels of algebraic thinking use symbols and algebraic relationships in tasks within the context of geometry and measurement?

RQ 3) How do students structure patterns and formulate generalizations?

RQ 4) How does the use of multiple representations differ according to algebraic thinking levels?

RQ 5) What relationship is observed between algebraic thinking levels and reasoning processes in geometric contexts?

4. Methodology

4.1. Research Design

This study was conducted using a basic qualitative research design in order to examine how eighth-grade students use algebraic thinking in geometry and measurement tasks and to identify the conceptual difficulties they encounter in this process (Merriam & Tisdell, 2016). In the analysis process, students' use of symbolic expressions, generalization strategies, and transitions among representations was taken into account. The clinical interview method was employed to enable an in-depth examination of students' cognitive processes (Ginsburg, 1997; Goldin, 2000).

4.2. Participants

The study was conducted with eighth-grade students attending a public middle school with a middle socioeconomic profile. The selection of the participants was carried out in two stages. In the first stage, the Algebraic Thinking Level Assessment and the Van Hiele Geometric Thinking Level Test were administered to 187 students. In the second stage, only students identified as being at Van Hiele Level 3 (analytic level) were selected. This preference enabled the analysis to focus on components of algebraic thinking by controlling for differences that might stem from students' geometric thinking levels.

Among the 24 students at this level, two students from each algebraic thinking level (high, medium, and low) were selected based on their scores, resulting in a total of six participants. To ensure confidentiality, pseudonyms were assigned to the participants. The algebraic thinking levels of the participants and the codes used in the analysis process are presented in Table 1.

Table 1
Algebraic Thinking Levels of Participants

Algebraic thinking level	*Participants*
High	Sahra (S_y), Murat (M_y)
Medium	Esra (E_o), Uras (U_o)
Low	Defne (D_d), Alp (A_d)

Note. Pseudonyms were used to protect participants' identities.

4.3. Data Collection

Data were obtained from three sources: (i) the Algebraic Thinking Level Assessment, (ii) the Van Hiele Geometric Thinking Level Test, and (iii) clinical interviews. The data collection process was carried out in two stages. In the first stage, the tests were administered, and in the second stage, clinical interviews were conducted with the selected students.

4.3.1. Algebraic thinking level assessment

The Algebraic Thinking Level Assessment, developed by the researchers, consists of 11 open-ended questions. The test covers the components of symbols and algebraic relationships, patterns and generalization, and the use of multiple representations. The distribution of the test items according to these components is presented in Table 2.

Table 2

Distribution of the Items of the Algebraic Thinking Level Assessment According to Theoretical Components

Component	*Number of Item–Component Matches*	*Total Score*
Symbols and Algebraic Relationships	6	39
Patterns and Generalization	4	28
The Use of Multiple Representations	4	33

Note. Multiple Representations was translated as The Use of Multiple Representations to ensure terminological consistency.

As shown in Table 2, a total of six items (39 points) are included under the component of symbols and algebraic relationships. These items address contexts such as simplifying algebraic expressions, solving equations, calculating the perimeter and area of a rectangle whose side lengths are represented algebraically, substituting values into algebraic expressions with two variables, and establishing functional relationships between variables. For example, the following item, which requires transforming a verbal situation into an algebraic representation, primarily belongs to the component of symbols and algebraic relationships; however, it is also related to the component of patterns and generalization due to its focus on establishing a functional relationship between variables.

> Add 3 to twice a number x. If you square the number you obtain and add 10, you obtain y. In that case, can you write an algebraic expression that shows the relationship between x and y?

Under the component of the use of multiple representations, four items (33 points) are included. These items address contexts such as generating an algebraic expression from a table representation, drawing a graph from tabular data, transforming a given equation into a graphical representation, and examining consistency across representations. For example, the following item is related to both the use of multiple representations and the component of symbols and algebraic relationships, as it requires analyzing the structural properties of linear relationships through representation transformation.

> Based on the tables given below, write the equation that best describes the relationship between x and y for each situation. Explain your solution in detail.
>
> a) $y = 5x + 10$

x						
y						

> b) $y = 3x$

x						
y						

c) $y = 5x^2$

x						
y						

Examine the tables and the equations.

(i) Which relationship is linear, and which is not? Why?

(ii) Based on the data in the tables or the equations, how do you decide whether a relationship is linear? Explain in detail.

Under the component of patterns and generalization, four items (28 points) are included. These items address contexts such as formulating near- and far-term generalizations in numerical patterns, transitioning from verbal representations to algebraic representations, interpreting algebraic representations, and formulating functional generalizations by systematically applying a given rule to different values. For example, the following item addresses the component of patterns and generalization by requiring students to generalize a rule for different inputs; at the same time, because it involves transitions between tabular and graphical representations, it also addresses the component of the use of multiple representations.

Given $\mathrm{F} = \frac{7}{5}\mathrm{C} + 42$, fill in the blanks below.

C:	10	20	30	40	50
F:	56	70		98	

Based on this, how would you explain the relationship between F and C using a graph?

An analytic scoring rubric was used to evaluate the open-ended items. A criterion-referenced assessment approach was adopted, and students' performance was evaluated according to predetermined criteria (Nitko & Brookhart, 2014; Popham, 2017). Evaluations were conducted on a 100-point scale, and based on score ranges, students were classified as high (75–100), medium (50–74), and low (0–49).

To ensure the content validity of the test, the opinions of two field experts and one middle school mathematics teacher were obtained, and necessary revisions were made accordingly. The pilot implementation was conducted with 90 students who shared similar characteristics with the main study group; in addition, a preliminary implementation was carried out with three students to examine the clarity of the items. The scoring was performed by two independent raters, disagreements were resolved through discussion, and inter-rater agreement was calculated as 90% (Miles & Huberman, 1994).

4.3.2. Van Hiele geometric thinking level test

In order to determine students' geometric thinking levels, the Turkish version of the test based on the Van Hiele model was used (Usiskin, 1982; Van Hiele, 1986). This test was used solely as a filtering instrument during the sampling process.

4.3.3. Clinical interviews

Clinical interviews were conducted with the six selected students. The interview protocol consisted of a total of 12 questions addressing the components of symbols and algebraic relationships (5 questions), patterns and generalization (3 questions), and the use of multiple representations (4 questions).

Within the component of symbols and algebraic relationships, students' interpretations of variables, their construction of algebraic expressions, and their interpretations of relationships among quantities were examined. For example, through the following question, students' abilities to combine algebraic expressions, interpret the variable as a generalized quantity, and distinguish the functions of mathematical symbols across different contexts were investigated.

Given that the side lengths of triangle ABC are $|AB| = 3x - 2$, $|BC| = 2x - 1$ and $|AC| = x + 5$:

(a) Find the perimeter of triangle ABC. Explain how you found it.

(b) If x increases, how does the perimeter of triangle ABC change? Explain in detail.

(c) In the problem, what does the letter x represent, and what does the letter A, which represents a vertex of the triangle, mean? Are they used in the same way? Explain in detail.

Item (c) of the sample question is particularly intended to reveal students' level of symbolic awareness, as it requires students to compare the use of letters as variables in algebra with their use as object labels in geometry. In this respect, although the question belongs to the component of symbols and algebraic relationships, it is also related to the component of patterns and generalization because it involves conceptual generalization.

Within the component of the use of multiple representations, students' processes of establishing relationships among geometric, algebraic, and verbal representations and producing different forms of representation were examined. For example, the following question aims to reveal students' abilities to express a geometric relationship through different representations and to interpret the concept of equivalence.

Points A, B, and C are three collinear points such that $|AB| = |BC|$. How and in how many different ways can you represent these points on the coordinate plane? Explain your solution.

Within the component of patterns and generalization, students' processes of distinguishing between varying and invariant quantities in geometric structures, formulating far-term generalizations from near terms, and transforming the resulting relationship into a symbolic model in terms of the variable n were examined. Accordingly, the questions were designed to require students to construct functional relationships among quantities through polygons and situations involving geometric growth.

For example, in the following question, the student is expected to derive a general model based on specific cases:

What is the relationship between one interior angle and one exterior angle of a triangle? Explain.

- How many interior angles and how many exterior angles does a triangle have?
- What is the sum of all interior angles and all exterior angles of a triangle? Why?
- What is the sum of the measures of the interior and exterior angles of a quadrilateral? Why?
- What is the sum of the measures of the interior and exterior angles of a pentagon, a hexagon, and an n-sided polygon? Explain.

Through such questions, students' abilities to formulate generalizations based on numerical examples, justify their generalizations (e.g., through the interior–exterior angle relationship), interpret the variable as a quantity representing the number of sides of a polygon, and express the derived rule symbolically were examined. Three sessions were conducted with each student, each lasting approximately 50–60 minutes. The interviews were carried out in an appropriate setting, and audio and video recordings were obtained and subsequently transcribed into written form.

4.4. Data Analysis

In the analysis of the data, thematic analysis and content analysis were used in combination. Within the scope of thematic analysis, the main themes were predetermined based on the theoretical framework of the study. Accordingly, three main themes—"Symbols and Algebraic Relationships," "Patterns and Generalization," and "The Use of Multiple Representations"—were structured as the initial analytic framework through a deductive approach. During the content analysis process, clinical interview data were examined not on an item-by-item basis but on a component-by component basis. Questions addressing more than one component were evaluated separately in relation to each relevant component indicator.

The coding process was conducted using both deductive and inductive approaches. The interview transcripts were independently coded by the researcher and a field expert, and disagreements were resolved through discussion. Inter-coder reliability was calculated using the

Miles and Huberman (1994) formula and was found to be 90%. To illustrate how the analysis process was conducted, a data excerpt related to the component of patterns and generalization is presented below.

U_o, in a task concerning the exterior angles of regular polygons, stated that the sum of the exterior angles of a pentagon and a hexagon is 360°, and when asked to generalize for an n-sided polygon, expressed the following:

> We found one exterior angle and multiplied it by the number of sides. Then it would be one exterior angle times n.

In the analysis process, this statement was first coded at the solution-step level. It was determined that the student tended to use a variable based on numerical examples; however, the student was not able to transform the generalization into a formal and systematic algebraic expression. It was also observed that although the same student symbolically expressed the exterior angles of a non-regular polygon, the student was unable to transform these expressions into a general model.

The coding process was structured in the form of solution step → code → subtheme → main theme. A summary of this process is presented in Table 3.

Table 3

Coding Example Related to U_o's Solution Process

Data Excerpt	*Code*	*Subtheme*	*Main Theme*
"We found one exterior angle and multiplied it by the number of sides."	Attempt to generalize through multiplication	Attempt at structural generalization	Patterns and Generalization
Expressing the exterior angles as 180 – a and 180 − b	Partial symbolization	Partial symbolization in generalization	Patterns and Generalization
Inability to transform the expressions into a general model	Limitation in generalization	Structural deficiency	Patterns and Generalization

Note. Coding was initiated at the solution-step level. In some data excerpts, the use of symbolic expressions was evaluated as part of the process of producing generalizations; therefore, symbolization behaviors were coded not as independent instances of representation use but within a subtheme reflecting the nature of generalization.

During the coding process, cross-coding was applied in cases where certain solution steps simultaneously reflected more than one cognitive component. In this approach, each data excerpt was first labeled with a micro-code describing the action performed by the student in the solution process; subsequently, this code was grouped under the relevant subtheme(s) and, together with its justification, linked to the corresponding main theme(s) in the theoretical framework.

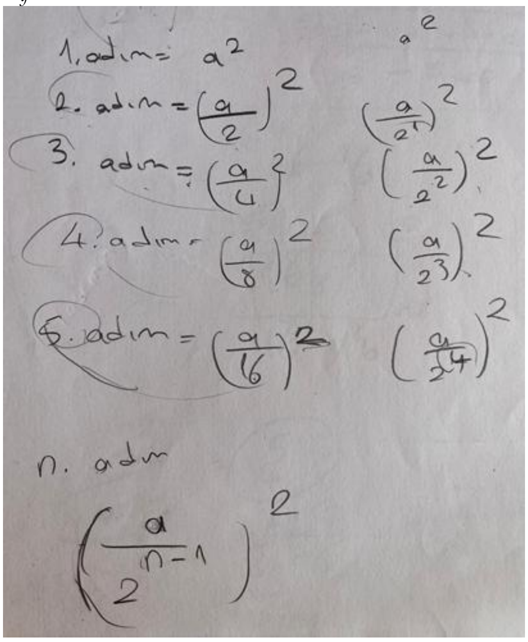
For example, in S_y's area pattern task, the statement "It will be $a/(2n-1)$, and then its square will be taken" was initially coded as "constructing a formal algebraic model." This code was associated, on the one hand, with the theme of Patterns and Generalization due to its role in establishing a functional relationship between the step number (n) and the quantity (area) and producing a far-term generalization; on the other hand, because the student transformed a variable expression into a meaningful algebraic structure and produced a symbolic model, it was also linked to the theme of Symbols and Algebraic Relationships.

Similarly, U_o's statement in the context of exterior angles of regular polygons—"We found one exterior angle and multiplied it by the number of sides. Then it would be one exterior angle times n"—was coded as "attempt at structural generalization" under Patterns and Generalization, as it reflected a tendency toward generalization. However, the student's limitation in transforming the symbolic expression into a general model and in using the variable as a relational tool was additionally noted with respect to the quality of symbolization. Accordingly, some examples were reported under more than one theme in the findings section, while preserving their contexts, in order to demonstrate the interrelated nature of components such as generalization and symbolic meaning-making.

In the pattern task, Sy first constructed a systematic list showing the relationship between the step numbers and the corresponding area values in order to determine the far term value (see Figure 1).

Figure 1

S_y's Solution



The student searched for a functional relationship between the step number and the area based on near-term cases and provided the following explanation:

> Researcher: What will we do for the nth step now?
>
> S_y: We need to find the connection between these two… In the 5th step it's 4, in the 4th step it's 3, in the 3rd step it's 2… It decreases by 1 each time. It will be a divided by $(2n - 1)$, and then we will take its square.

This data excerpt was first coded at the solution-step level during the analysis process. It was determined that the student:

- Established a systematic relationship between the step number and the pattern structure,
- Interpreted the variable (n) as a quantity representing the step number,
- Reached an algebraic model based on the contextual relationship.

As a result of the coding process, the structure presented in Table 4 was constructed:

Table 4

Coding Example Related to S_y's Solution Process

Data Excerpt	*Code*	*Subtheme*	*Main Theme*
Constructing a list showing the relationship between the step number and the area	Search for a functional relationship	Structural pattern analysis	Patterns and Generalization
"It decreases by 1 each time."	Establishing a variable-based relationship	Functional generalization	Patterns and Generalization
Producing the expression "$\frac{a}{(2^{n-1})}$, then its square"	Constructing a formal algebraic model	Symbolic modeling	Symbols and Algebraic Relationships

Note. During the coding process, the student's transition from contextual reasoning to a formal algebraic expression was evaluated as an indicator of a high level of algebraic thinking.

These two examples demonstrate that students' generalization processes differ across levels not only in terms of outcomes but also in terms of reasoning structure and the level of symbolization.

4.5. Validity and Reliability

To enhance the credibility of the study, data triangulation was ensured, and test data and clinical interview data were evaluated together. The interviews were recorded and transcribed into written form, and the findings were supported by direct quotations from participants. Conducting multiple sessions with each participant enabled a more in-depth examination of students' thinking processes. The coding process was carried out in a systematic and iterative manner; independent coding was conducted by the researcher and a field expert, and consensus was reached through comparison of the codes.

4.6. Ethical Considerations

Participation in the study was voluntary. Participants and their parents were informed about the research, and the necessary permissions were obtained. To ensure confidentiality, pseudonyms were used. The collected data were used solely for scientific purposes.

5. Findings

The data obtained from this study were analyzed to reveal how eighth-grade students used the components of algebraic thinking in geometry and measurement tasks. The findings are presented under three main themes: students' use of symbols and algebraic relationships, their transitions among representations, and their approaches to patterns and generalization.

In reporting the findings, students' solution processes were evaluated not only in terms of correct–incorrect outcomes but also with regard to their use of symbolic expressions, interpretations of variables, geometric meaning-making, and forms of reasoning. Direct quotations from clinical interviews and students' written work were used to support and substantiate the analytic findings.

5.1. Findings Related to Symbols and Algebraic Relationships

In this section, students' use of symbolic expressions in geometry and measurement tasks was examined at the solution-step level and explanations through content and thematic analysis of the clinical interview data. During the analysis process, students' ways of carrying out algebraic procedures, the meanings they attributed to symbols, and the relationships they established between geometric contexts and algebraic expressions were coded. The cognitive patterns that emerged during coding were organized under subthemes based on their shared characteristics.

The main theme, subthemes, and codes related to symbols and algebraic relationships are presented in Table 5.

Table 5
Theme, Subthemes, and Codes Related to Symbols and Algebraic Relationships

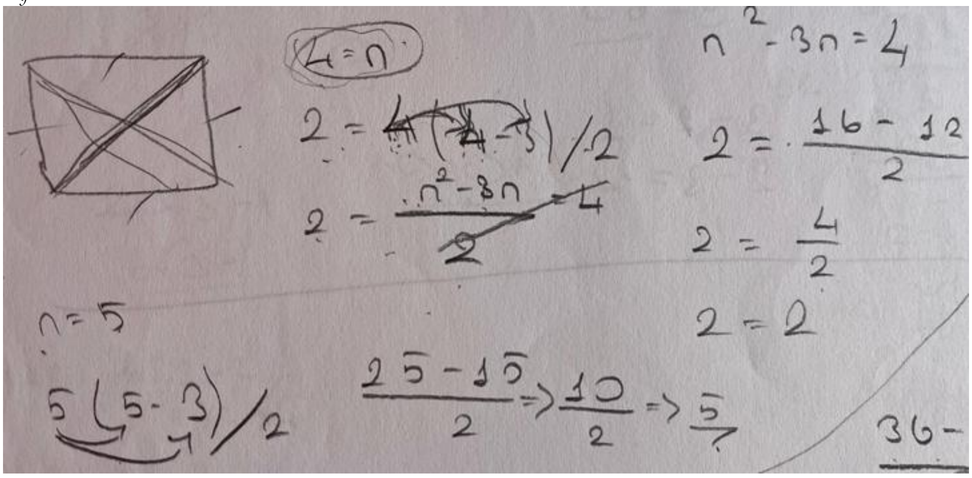
Main theme and subtheme	*Codes*
Symbols and Algebraic Relationships	
Procedural symbolic use	Assigning values to variables; operating with like terms; simplifying algebraic expressions; carrying out equation-solving steps
Interpreting the role of variables and symbols	Confusion between variable and unknown; attributing label meaning to symbols; interpreting coefficients as repeated quantities; difficulty distinguishing between expression and variable
Linking geometry and algebra	Transforming geometric quantities into algebraic expressions; performing geometric verification; attending to contextual conditions; interpreting the geometric meaning of a formula
Transition from verbal situation to algebraic model	Formulating equations from verbal statements; incomplete modeling of relationships; structural errors in equation construction; constructing a correct model but making errors in procedural steps

5.1.1. Procedural use of symbols

The most frequently observed behaviors in students' solution processes were assigning values to variables, operating with like terms, and directly applying a given formula. The emergence of these codes across all participants indicates that students generally approached geometric problem solving through procedural symbolic use.

In tasks requiring the simplification of algebraic expressions or the use of formulas, students tended to substitute numerical values for variables in order to reach a result. This suggests that symbolic expressions were primarily treated as tools serving the execution of computational procedures at the early stages of the solution process. However, the presence of procedural accuracy across all students does not necessarily imply that symbolic expressions were interpreted at the same conceptual level. An examination of students' explanations revealed that those with medium and high levels of algebraic thinking did not merely apply symbolic expressions but also used them for verification purposes by relating them to the geometric structure. For example, in Sy's solution, the student first drew the geometric figure and then verified the algebraic formula by referring back to that structure (see Figure 2).

Figure 2
S_y's Solution



In the solution process, the student stated, "Since I know that a quadrilateral has two diagonals, I set the formula equal to two… I substituted four for n and checked whether it was correct." Such solution processes indicate that symbolic expressions can be used not only as procedural tools but also as structures for testing geometric relationships.

5.1.2. Interpreting the roles of the variables and the symbols

One prominent pattern that emerged during the analysis was the variation in students' conceptions of variables. It was observed that some students tended to interpret literal expressions primarily as unknowns and had difficulty distinguishing among the roles of variables, labels, and unknowns. Dd's statement reflects this situation: "It cannot be an unknown because it is written in uppercase . . . if it were lowercase, it would be a variable." Such explanations indicate that the mathematical function of symbols was interpreted based on formal features rather than contextual meaning. Similarly, some students were found to interpret coefficients as "repeated quantities" and to treat the algebraic expression as if it represented a single unknown in its entirety.

In contrast, students who were able to interpret the variable as a generalized quantity used symbolic expressions more flexibly for purposes of justification and establishing relationships. However, among students who regarded the variable solely as an unknown, symbolic use tended to remain at a procedural level.

5.1.3. Linking geometry and algebra

The findings indicate that the geometric context constituted an important reference point in students' processes of making sense of symbolic expressions. In tasks related to the perimeter of a triangle, students directly associated algebraic expressions with side lengths and interpreted the resulting algebraic expression as representing a geometric quantity.

For example, in S_y's solution, explaining the expression $6x + 3$ as "the perimeter of the triangle" demonstrates that the symbolic result was successfully linked to its geometric meaning.

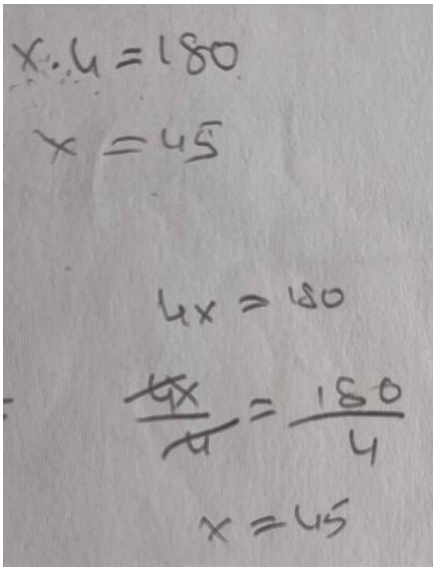
It was also observed that students at medium and high levels of algebraic thinking checked whether the values they assigned to the variable satisfied conditions such as the triangle inequality or the requirement that lengths should not be negative. This indicates that algebraic procedures were not carried out independently of the context. Rather, students evaluated symbolic operations within a geometric meaning framework.

5.1.4. Transition from a verbal situation to an algebraic model

In tasks requiring the transformation of verbally presented geometric relationships into algebraic models, students exhibited different modeling approaches. Students with a high level of algebraic thinking were able to translate the verbal statement directly into an equation and construct a correct model. In contrast, some students were unable to incorporate the supplementary relationship embedded in the verbal statement into the model and represented the relationship incompletely.

For instance, A_d modeled the situation in which four times the measure of an angle is equal to the measure of its supplement as $4x = 180$ (see Figure 3).

Figure 3
A_d's Solution



Although some students reasoned through geometric representations, the construction of incorrect algebraic models indicates that conceptual discontinuities may occur during transitions between representational levels. In general, a guess-and-check strategy was found to be dominant among students at the low algebraic thinking level, whereas the tendency to construct algebraic models became more pronounced among students at medium and high levels.

When the findings related to symbols and algebraic relationships are considered collectively, it becomes evident that although students were able to use algebraic symbols in geometry-based problems, the quality of this use varied according to their level of algebraic thinking. Procedural use of symbols—such as assigning values to variables and carrying out algebraic operations—emerged as a common feature across all students. However, clear differences were observed in processes involving the interpretation of symbols, understanding the roles of variables, and transforming verbal relationships into algebraic models.

In particular, students who were able to interpret the variable as a generalized quantity used symbolic expressions to verify and justify relationships by linking them to the geometric context. In contrast, among students who viewed the variable solely as an unknown, symbolic use largely remained at a procedural level. Moreover, as the relationship established between symbolic and geometric representations strengthened, students demonstrated more coherent reasoning in equation construction and modeling processes. These findings suggest that the ability to perform symbolic operations alone does not necessarily indicate the development of algebraic thinking; rather, making sense of symbols and integrating them with contextual relationships plays a decisive role.

5.2. Findings Related to the Use of Multiple Representations

In this section, students' use of representations in geometry and measurement tasks was examined within the framework of representational diversity, transitions among representations, attention to contextual conditions, levels of representation interpretations and difficulties encountered in representation use. The data obtained from the clinical interviews were analyzed through content and thematic analysis; students' ways of producing different types of representations, transitioning among them, and preserving mathematical meaning were examined.

The theme, subthemes, and codes that emerged from the coding process are presented in Table 6.

Table 6
Theme, Subthemes, and Codes Related to the Use of Multiple Representations

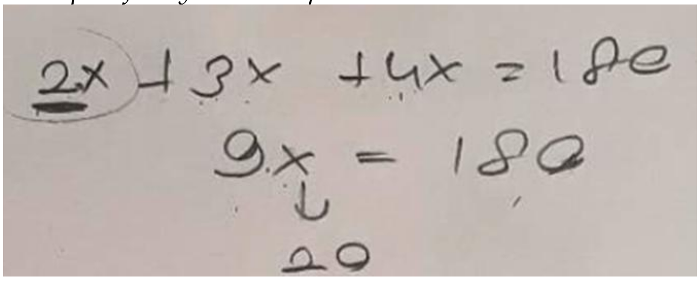
Main theme and subtheme	*Codes*
The Use of Representations	
Representational diversity	Using symbolic/algebraic representations; constructing visual (figure-based) representations; producing numerical representations; representing through verbal explanations
Transitions among representations	Transition from verbal to numerical; from numerical to symbolic; from symbolic to visual; from visual to symbolic
Attending to contextual conditions	Tendency toward a prototypical representation; revising representation upon recognizing contextual constraints; restructuring toward an appropriate representation
Level of representation interpretation	Using literal expressions as labels; interpreting the variable as a multiplier/scale factor; producing contextual justification
Difficulties in representation use	Generating solutions through trial and error; misconception of linearity; transferring area as a rule; loss of meaning during representation transformation

5.2.1. Representational diversity

The findings indicate that students used symbolic/algebraic, visual, numerical, and verbal representations either individually or in combination during problem-solving processes. However, the emergence of representational diversity was largely determined by the task context rather than by students' cognitive preferences.

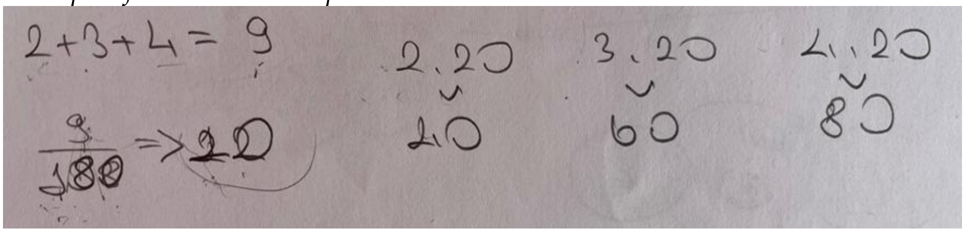
In an angle problem involving direct proportion, some students began the solution process directly with a symbolic representation and modeled the relationship in the form $2x + 3x + 4x = 180$ (see Figure 4).

Figure 4
Example of a Symbolic Representation



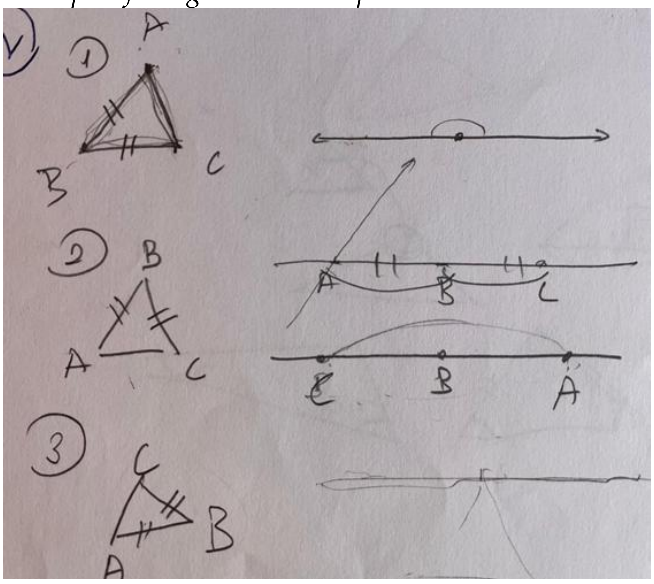
These students used the variable as a tool representing the multiplicative relationship among quantities. In contrast, some students began the solution process with a numerical representation and attempted to reach a result by trying appropriate values (see Figure 5).

Figure 5
Example of a Numerical Representation



In the task involving the statement "A, B, and C are three collinear points," the production of representations occurred predominantly at the visual level. Most students initially constructed a figure-based representation, and the algebraic expression was generally used as a supporting explanatory element accompanying the drawing (see Figure 6).

Figure 6
Example of a Figure-Based Representation

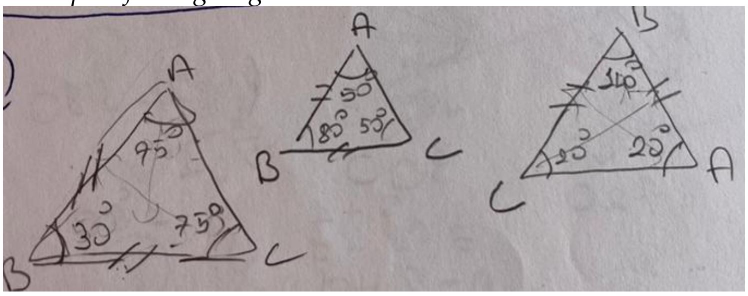


In the task requiring an explanation of the area of a triangle, the dominant representational pattern emerged in the sequence of verbal rule → figure-based representation → numerical example. This pattern suggests that students primarily used representations to explain a known rule rather than to explore or construct the underlying mathematical relationship.

5.2.2. Transitions among representations: from procedural focus to meaning-oriented reasoning

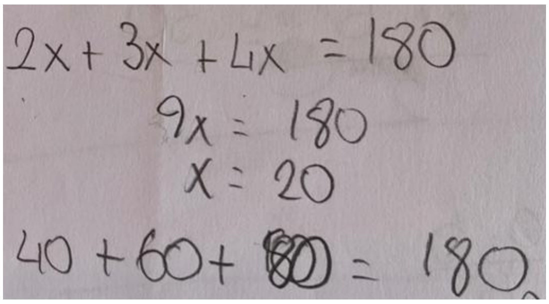
Two main cognitive patterns were identified in students' transitions among representations. The first pattern, observed particularly among students with a low level of algebraic thinking, involved moving from a verbal representation to numerical trial. Although these students were able to articulate the proportional relationship verbally, they progressed in the solution process by trying numerical values and experienced difficulty transitioning to a symbolic model (see Figure 7). This finding indicates that changing representations alone does not necessarily lead to conceptual advancement.

Figure 7
Example of Assigning Numerical Values



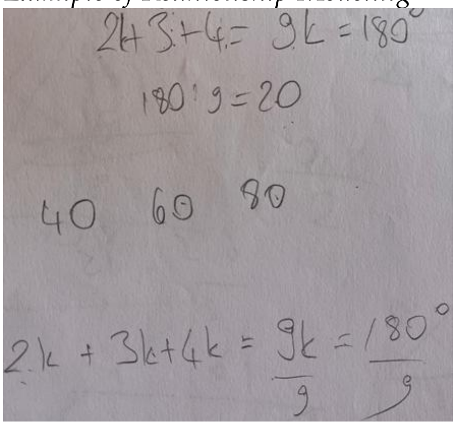
The second pattern, observed among students at medium and high levels of algebraic thinking, was an early symbolic modeling process. For example, in Eo's solution, the constructed algebraic model structured the direct proportion through a multiplicative ("times") relationship, thereby preserving meaning across representations (see Figure 8).

Figure 8
E_o's Solution



Similarly, some students transformed their numerical reasoning into symbolic expressions, indicating that the transition between representations served not merely to generate procedures but to model the underlying relationship (see Figure 9).

Figure 9
Example of Relationship Modeling

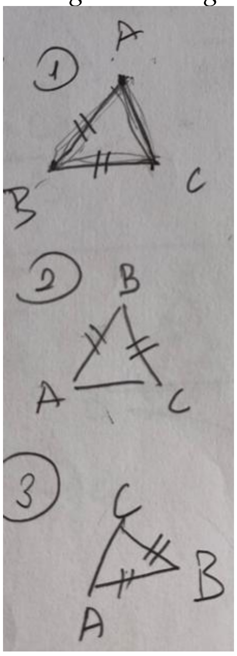


This finding indicates that the quality of transitions among representations is related less to the type of representation used and more to students' understanding of the role of the variable.

5.2.3. Attending to contextual conditions: from prototypical representation to revision

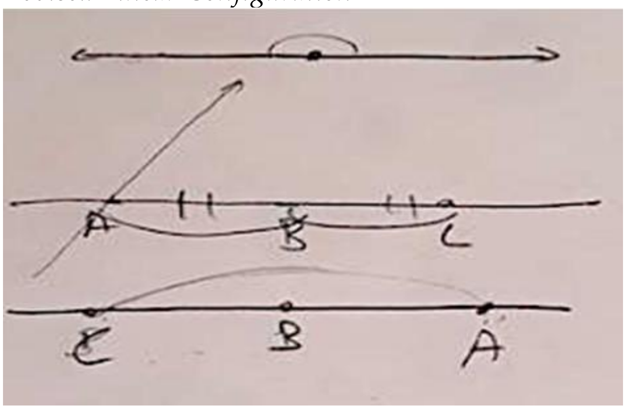
One of the most prominent patterns in representation use was students' tendency to rely initially on prototypical representations. In the task "A, B, and C are three collinear points," many students automatically associated the given relationship with a triangular structure and produced a triangle drawing (see Figure 10).

Figure 10
Triangle Drawing



This finding suggests that, at the initial stage, representation production was guided more by familiar visual schemas than by the mathematical conditions of the task. However, some students, after recognizing the condition of collinearity, revised their representations and positioned the points on the same line (see Figure 11). The restructuring of the representation indicates that the student did not merely produce a drawing but was able to establish a relationship between the representation and the mathematical condition.

Figure 11
Revised Linear Configuration



5.2.4. Level of representation interpretation

In the task requiring an explanation of the area of a triangle, the majority of students correctly stated that the area is obtained as one half of the product of the length of a base and the length of the altitude corresponding to that base. However, students' explanations largely remained at a procedural level. Although students were able to state the formula correctly, they provided limited justification for the underlying geometric meaning of this relationship and offered little explanation regarding the relationship between the quantities involved. Literal expressions were mostly used in a label-like sense; for instance, the variable a was used to denote the length of a base and the variable b was used to denote the length of the corresponding altitude. This pattern of use suggests that the role of variables as representing generalized quantities had not yet been sufficiently structured in students' reasoning.

In contrast, some students extended the area relationship beyond the rectangle to other geometric structures such as the parallelogram, indicating that their use of representation reached a justificatory level. This finding suggests that the level of interpreting representations is associated less with knowledge of a formula and more with the capacity to establish relationships.

5.2.5. Difficulties encountered in the use of representations

The findings indicate that students' difficulties related to representation use were concentrated not on producing representations per se, but on preserving mathematical meaning during transitions among representations. In most cases, students were able to express a problem verbally, numerically, or visually; however, they experienced difficulty maintaining the consistency of the same relationship across different representations. In particular:

- Addressing direct proportion through trial and error,
- Overlooking the condition of linearity,
- Treating the area formula merely as a rule to be stated,
- Failing to distinguish between labels and variables in literal expressions

all indicated that coordination among representations remained limited.

When the findings related to representation use are considered collectively, it becomes clear that although students were generally capable of producing different types of representations, what is decisive for algebraic thinking is not representational diversity itself but the ability to establish continuity of meaning across representations. While representation production was often procedural or prototype-based, recognizing contextual conditions and revising representations emerged as indicators of more advanced algebraic thinking. These findings suggest that representation, symbolic meaning-making, and generalization processes form an integrated cognitive structure rather than functioning independently.

5.3. Patterns and Generalization

In this section, the component of patterns and generalization was examined through two tasks administrated within the scope of the clinical interviews (the sum of exterior angles in polygons and the square/area pattern task). The analyses revealed that students displayed a differentiated cognitive progression in their generalization processes, moving from iterative reasoning based on specific cases toward expressing relationships among quantities through generalized algebraic structures.

The findings indicate that, particularly in the initial stages, students approached generalization by examining successive cases and engaging in example-based reasoning. However, for far-term cases, the quality of generalization diverged according to students' ability to use the variable as a representation of a generalized quantity. The theme, subthemes, and codes related to patterns and generalization processes are presented in Table 7.

Table 7
Theme, Subthemes, and Codes Related to Patterns and Generalization Processes

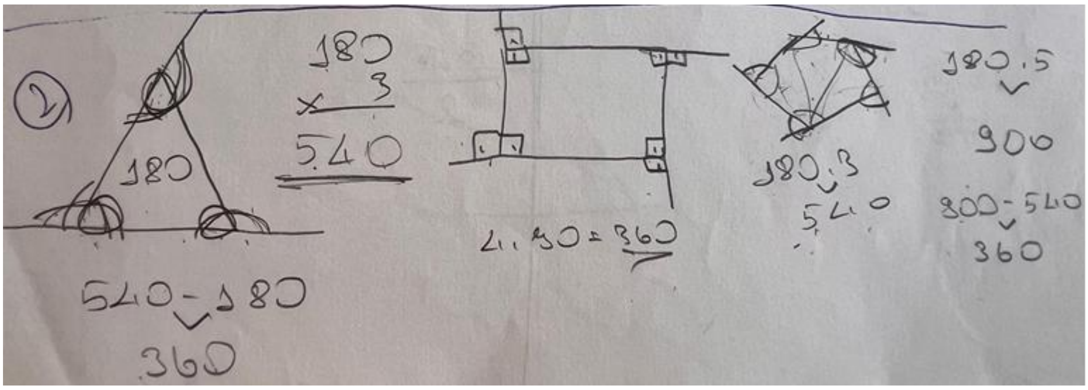
Main theme and subtheme	*Codes*
Patterns and Generalization Processes	
Iterative generalization at near terms	Examining successive cases; example-based reasoning; numerical verification; searching for relationships through specific cases; establishing iterative relationships
Structural (functional) generalization	Establishing a relationship between the step number and quantity; constructing a functional relationship; using variables; reaching a general model
Verification and revision of generalization	Verifying through substitution; testing in different cases; model revision; questioning the scope of the generalization
Types of reasoning	Iterative reasoning; contextual reasoning; proportional reasoning; extending to the whole
Difficulties in the generalization process	Misconceptions; rote use of rules; inability to use the variable as a relational tool; difficulty producing symbolic expressions

5.3.1. Iterative generalization at near terms

As shown in Table 7, the majority of students began the generalization process in the pattern tasks using an approach identified in the literature as iterative (recursive) generalization. In this process, students examined successive cases of the pattern, generated each new step based on the previous one, and attempted to make sense of the emerging regularity through local examples.

In the task concerning the sum of the measures of the exterior angles of polygons, students examined successive polygons such as triangles, quadrilaterals, and pentagons, calculated the sum of the measures of the exterior angles for each case, and compared the results in search of a common regularity. This approach indicates that, at the initial stage of generalization, students focused more on similarities across cases than on the underlying relational structure (see Figure 12).

Figure 12
S_y's solution: Student drawing illustrating the extension of near terms of the pattern through successive cases



Similarly, in the area pattern task, students constructed the initial steps through drawings and examined how each step was transformed from the previous one. In this process, generalization was based on recognizing the multiplicative or divisional change between steps; however, in most cases, this relationship had not yet been associated with a variable representing the step number (see Figures 13 and 14).

Figure 13
D_d – The student's near-term solution in the area pattern constructed through a recursive relationship

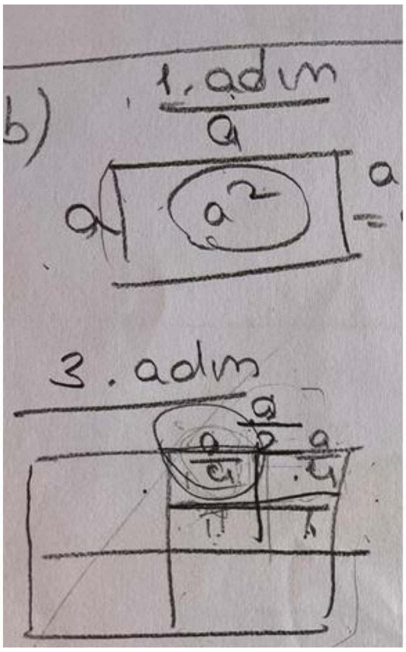
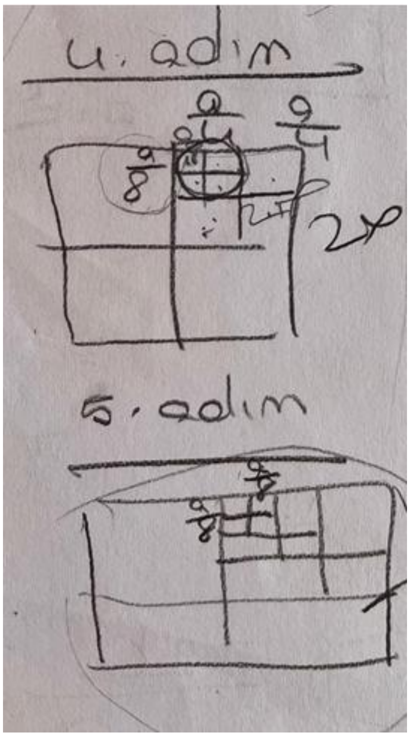


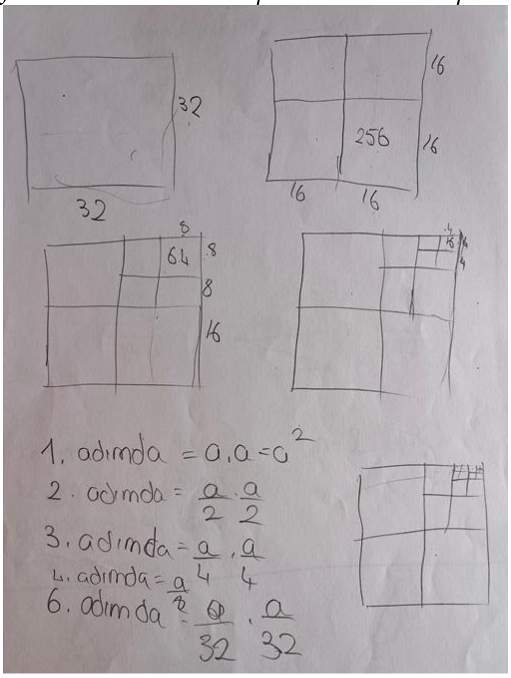
Figure 14
D_d – The student's near-term solution in the area pattern constructed through a recursive relationship



This finding indicates that although recursive generalization enables students to notice regularities within a pattern, it is not sufficient on its own for transitioning to a generalized structure. Indeed, although some students recognized the regularity in the numerical sequence (e.g., 2, 4, 8, 16), they initially experienced difficulty expressing this pattern as a functional relationship with the step number (see Figure 15).

Figure 15

E_0 – The student's near-term work in which the numerical regularity in the pattern was recognized, yet a functional relationship between the step number and the quantity was not established



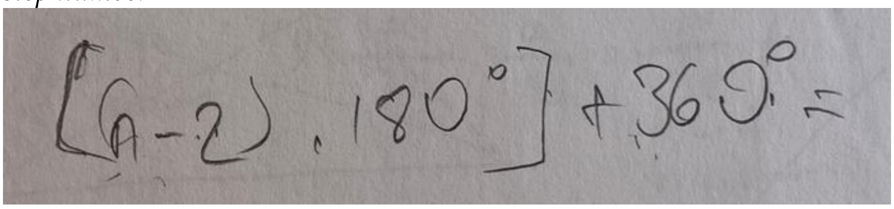
5.3.2. Structural (functional) generalization: Modeling the relationship between quantities

At the stage of far-term generalization, cognitive differentiation among students became more evident. Students at the medium and high levels of algebraic thinking moved beyond recursive relationships and began to address the pattern at the level of functional generalization. In this process, generalization involved not merely extending specific cases, but establishing an invariant relationship between the step number and the quantity under consideration.

In the task concerning the sum of the measures of the exterior angles, some students justified the constancy of the exterior angle sum not only through examples but also through total angle relationships inherent in polygonal structures. Such explanations indicate that generalization was constructed through contextual reasoning and that the student used geometric properties as the basis for generalization (see Figure 16).

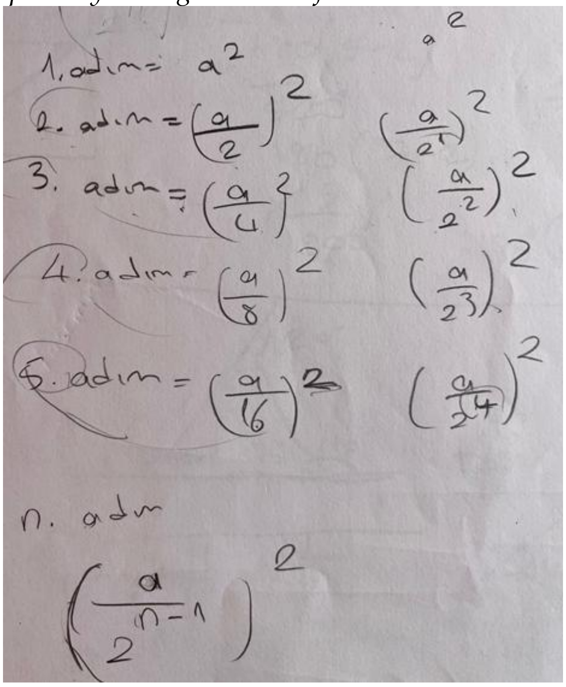
Figure 16

U_0 – Student work illustrating the association of the numerical regularity emerging in the pattern with the step number



In the area pattern task, functional generalization was observed more explicitly. Some students recognized that there was an exponential relationship between the step number and the resulting area and attempted to express this relationship using a variable. For example, one student's explanation that "in the nth step, the exponent is one less than the step number" indicates that the generalization shifted from a recursive relationship to a functional model (see Figure 17).

Figure 17
M_y – Student model illustrating the generalization of the relationship between the step number and the quantity through the use of a variable



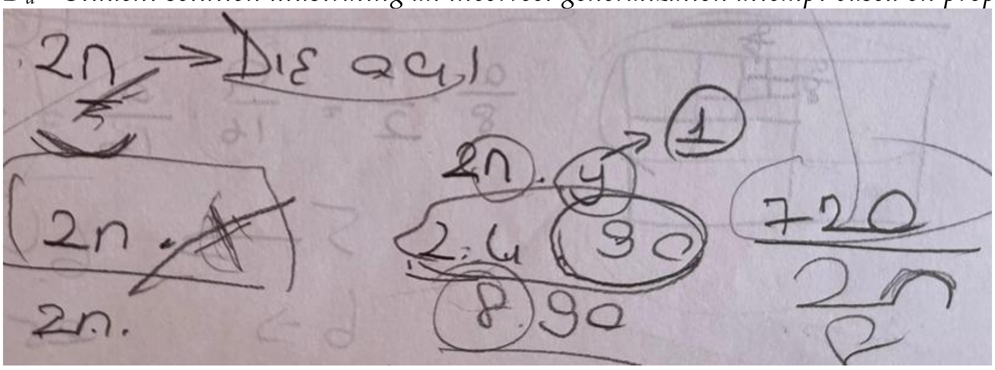
This finding indicates that functional generalization emerges when the student is able to treat the variable not merely as an unknown, but as a representation of a generalized quantity.

5.3.3. Erroneous generalization and partial symbolization

The analyses showed that during far-term generalization, some students tended to generalize surface-level numerical regularities observed in the pattern rather than establishing the structural relationship between quantities. This situation is described in the literature as partial generalization.

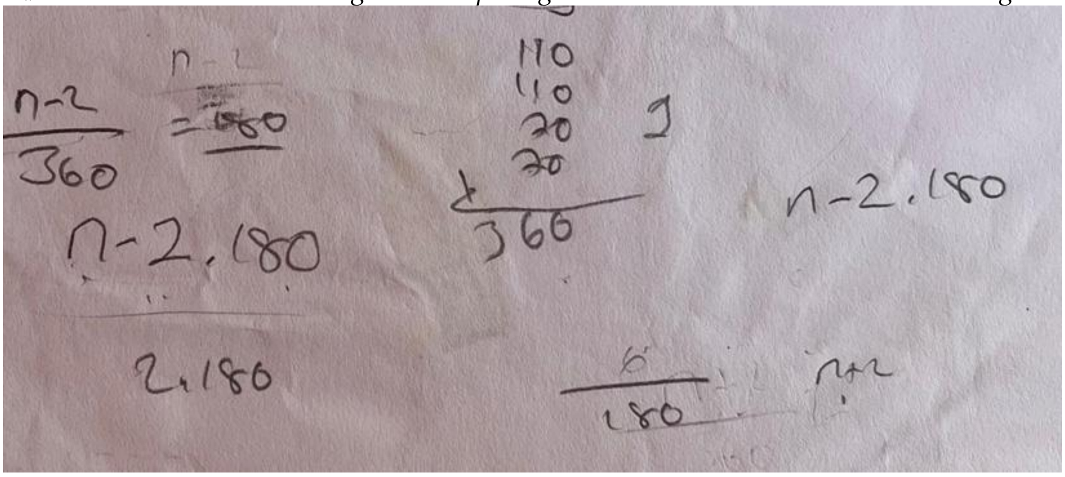
For example, some students related the sum of exterior angles linearly to the number of sides and produced expressions such as 2n; however, they were unable to justify which quantity this expression represented. In such cases, although a symbolic expression was produced, no meaningful connection was established between the symbol and the mathematical structure it was intended to represent (see Figure 18).

Figure 18
D_d – Student solution illustrating an incorrect generalization attempt based on proportional reasoning



Similarly, some students relied on recalling previously learned formulas during the generalization process, indicating that generalization was carried out on the basis of procedural knowledge rather than relational reasoning (see Figure 19).

Figure 19
A_d – Student work illustrating an attempt at generalization based on a memorized algebraic expression



5.3.4. Conceptual and symbolic difficulties encountered in the generalization process

The findings indicate that the difficulties experienced during the generalization process did not stem solely from algebraic notation errors; incomplete conceptualizations of geometric concepts also directly affected the construction of generalization. In particular, misconceptions related to the concept of an exterior angle led students to base their patterns on incorrect quantitative relationships. However, the more prevalent difficulty was students' inability to express the regularity they had noticed through a variable representing the step number. In many cases, students were able to follow the pattern; yet they struggled at the stage of transforming this pattern into a generalized algebraic model.

Overall, students were able to track the pattern in near terms through recursive reasoning; however, success in far-term generalization was directly related to representing the relationship between quantities through a variable and justifying that relationship. This finding is consistent with the processes of symbolic meaning-making and representation use discussed in previous sections.

When the findings of the study are considered as a whole, it is evident that eighth-grade students are able to use components of algebraic thinking in geometry and measurement tasks; however, the quality of this use varies substantially across processes of symbolic meaning-making, representation coordination, and generalization. Within the theme of symbols and algebraic relationships, most students were able to use algebraic expressions at a procedural level; nevertheless, differences emerged in their ability to interpret the quantitative relationships represented by symbols and to conceptualize the variable as a generalized quantity. This differentiation became more pronounced in representation use; although students were generally able to produce different types of representations, they experienced difficulty in maintaining mathematical meaning during transformations between representations.

The findings related to patterns and generalization processes indicate that the determining factor in the development of algebraic thinking is not merely recognizing a pattern, but structuring the relationship between quantities through a variable. Although most students were able to successfully extend near-term cases through recursive reasoning, reaching the level of functional generalization was found to be closely associated with the conceptual coherence demonstrated in representation use and symbolic meaning-making processes. Students who were able to interpret the variable as a carrier of relationships made more consistent transitions between representations and were able to justify their generalizations; in contrast, among students who treated the variable primarily as an unknown or a label, generalization remained at the level of recursive procedures.

These results indicate that the components of algebraic thinking—symbolic procedural skills, representation use, and generalization processes—form an integrated cognitive structure that supports one another. Therefore, evaluating students' algebraic thinking performance solely on the basis of reaching the correct result is insufficient; a more informative approach involves assessing their ability to make sense of mathematical relationships through multiple representations, to use the variable as a generalized quantity, and to justify the models they construct.

6. Discussion

The findings of this study indicate that eighth-grade students are able to use components of algebraic thinking at varying levels in geometry and measurement tasks; however, the quality of this use differs markedly across processes involving symbolic expressions, representation use, transitions between representations, and generalization. Beyond differences in performance among students, the findings reveal cognitive distinctions in how mathematical relationships are constructed, interpreted, and generalized. These results demonstrate that algebraic thinking cannot be reduced to symbolic procedural skills alone. Rather, they align with theoretical perspectives that conceptualize algebraic thinking as a multidimensional structure developed through the integrated functioning of structural, analytic, and functional processes (Kieran, 2022). At the same time, the present study contributes an additional perspective to the literature by showing that this integrated structure becomes particularly visible within geometric contexts through processes of establishing relationships among representations.

6.1. The Emergence of Algebraic Thinking in a Geometric Context

One of the notable findings of the study is that processes of algebraic thinking emerged naturally within geometry tasks (see Sections 5.1 and 5.2). In many cases, students used symbolic expressions not merely to carry out algebraic procedures, but to explain geometric relationships and to express relationships between quantities. This suggests that algebraic thinking is not a skill confined to a specific content domain; rather, it is a cross-domain mode of reasoning that emerges in various contexts where mathematical relationships are examined.

This finding is consistent with the work of Day et al. (2024), who demonstrate that multiplicative and proportional structures create conceptual continuity across algebraic, geometric, and statistical reasoning processes. However, the present study further indicates that such continuity does not arise spontaneously for students; rather, the tendency to employ symbolic representations becomes more pronounced in situations that require the analysis of geometric structure. In this sense, geometry tasks may be regarded not only as a domain in which algebraic thinking is applied, but also as a cognitive environment in which the idea of a variable and relational structures can be constructed.

Consistent with Grandau and Stephens's (2006) emphasis on structure sense, the findings of the present study show that students who were able to analyze geometric structures by decomposing them into their components were more successful in reaching functional generalization (see Section 5.3). This may be interpreted as evidence that structural analysis helps students organize relevant information, reduce unnecessary cognitive demands, and construct more general mathematical relationships. This interpretation is also consistent with research on mathematical thinking and algebraic generalization, which emphasizes the importance of moving beyond procedural rule finding toward meaningful symbolic, relational, and generalizing activity (Kama et al., 2023; Kükey & Aslaner, 2023).

6.2. From Procedural Use to Structural Meaning-Making

The findings regarding formula use and the simplification of algebraic expressions (see Section 5.1.1) indicate that a considerable proportion of students were able to follow similar solution strategies at a procedural level. However, the differences that emerged in interpreting the geometric meaning of algebraic expressions suggest that procedural accuracy does not necessarily entail structural understanding. This distinction is consistent with research showing that students may carry out operations with algebraic expressions while still interpreting symbols operationally, as object labels, or as unknowns rather than as representations of quantities and relationships (Bozkuş et al., 2025).

According to Kieran (2022), the development of algebraic thinking involves a transformation from the generalization of arithmetic operations to the recognition of structural relationships. In the present study, students at lower algebraic thinking levels tended to use symbols primarily as tools for computation, whereas students at higher algebraic thinking levels employed symbolic expressions to explain and verify geometric structures (see Section 5.1.3). These patterns reflect manifestations of the developmental transformation described in the literature within the learning environment. The lower cognitive demand associated with procedural strategies may help explain why such strategies are preferred, particularly in the early stages.

These results align with developmental models suggesting that the transition from arithmetic-based reasoning to generalized algebraic thinking is gradual (Sun et al., 2023). The present study further indicates that the critical factor in this transition is not merely symbolic proficiency, but the contextual interpretation of the relationships represented by symbols.

6.3. Variable Conceptions and Representation Coordination

The findings related to the interpretation of literal expressions (see Sections 5.1.2 and 5.2) indicate that a substantial proportion of students had difficulty distinguishing between the roles of variables, unknowns, and geometric labels. For instance, some students automatically interpreted uppercase letters as unknowns, while considering lowercase letters merely as tools representing multiplicative or scaled quantities. This suggests that the process of representation transformation, a fundamental component of algebraic thinking, is not yet fully developed. Similar difficulties have been reported in research on the transition from arithmetic to algebra, where students may interpret letters as object labels, unknowns, or fixed symbols rather than as representations of varying quantities and relationships (Bozkuş et al., 2025; Elbir & Özmen, 2024; Jupri et al., 2026). Radford (2003) reported that students initially make sense of symbolic expressions primarily through contextual and visual representations, with the transition to using symbolic formulas typically occurring during more advanced cognitive processes and far-term generalization. Similarly, Kieran (1981) found that students in early algebraic experiences struggle to distinguish among the different functions of variables (unknowns, variables, labels), which constrains their ability to generate algebraic generalizations.

When these studies are considered in relation to the findings of the present study, they highlight a cognitive pattern in which students' understanding of variables is not limited to their interpretation them as unknowns but also affects symbolic reasoning during generalization and transitions between representations. The results suggest that the development of students' ability to interpret the multiple roles of variables is closely linked not only to their symbolic manipulation skills but also to their capacity for representation coordination and generalization based on patterns. In this way, the empirical findings reported by Radford (2003) and Kieran (1981) are directly connected to the conceptual differentiations observed among students in this study.

Furthermore, Sarmasági et al. (2025) emphasize that algebraic thinking develops through the simultaneous operation of generalization, decomposition, and representation transformation processes. The present study also indicates that difficulties in transitioning between representations are largely attributable not to symbolic notation itself, but rather to the inability to preserve the mathematical relationships conveyed across representations. These findings highlight that representation coordination plays a central role in students' construction of mental models. In other words, students may be able to produce different types of representations; however, when continuity of meaning is not maintained, the process of algebraic generalization is disrupted.

6.4. The Developmental Nature of Pattern and Generalization Processes

The findings related to pattern tasks (see Section 5.3) indicate that students primarily approached the generalization process through recursive strategies. In near-term situations, students examined consecutive examples to detect regularities in the pattern. However, they struggled to translate this observed regularity into a functional model using the variable representing the step number.

Radford (2003) notes that in this process, students initially express relationships through gestures, verbal explanations, and visual representations, with symbolic formula generation typically emerging during far-term generalization. Similarly, researches (Blanton & Kaput, 2011; Kieran, 2022; Tanışlı & Yavuzsoy Köse, 2011) report that while students can detect patterns from numerical and visual examples, they face difficulties in transforming this awareness into a generalized model using variables. These findings suggest that generalization requires not only recognizing regularity but also modeling the quantitative relationships through variables while maintaining sufficient cognitive integration between symbolic meaning-making and representation use.

As noted by Apsari et al. (2020), geometric visualization plays a constitutive role in generalization. In the present study, students who were able to analyze geometric structures holistically were more likely to reach symbolic models. In contrast, students who merely extended successive cases tended to remain at a recursive level of generalization. This finding indicates that the transition to functional generalization is closely associated with the structuring of relational thinking.

6.5. Theoretical and Pedagogical Contributions of the Study

This study demonstrates that algebraic thinking at the middle school level cannot be explained solely through the development of symbolic procedural skills; rather, students' algebraic performance is fundamentally shaped by the interaction among symbolic meaning-making, representation coordination, and the ability to use the variable as a generalized quantity. The findings indicate that the emergence of algebraic thinking in geometric contexts is grounded not in a process of cross-domain transfer, but in the structuring of mathematical relationships across multiple representations. In this respect, the study extends existing perspectives on the development of algebraic thinking by foregrounding a representation-based cognitive coherence framework.

Accordingly, evaluating students' algebraic thinking performance solely in terms of arriving at the correct answer appears insufficient. A more informative approach involves assessing the extent to which students are able to make sense of mathematical relationships through multiple representations, use the variable as a generalized quantity, and justify the mathematical models they construct.

7. Conclusion and Implications

This study aimed to examine how eighth-grade students use the components of algebraic thinking in geometry and measurement tasks and to explore the reasoning patterns that emerge in this process. The findings indicate that although students are generally able to use algebraic expressions at a procedural level, they demonstrate marked differences in their ability to relate symbolic representations to geometric meanings, interpret the variable as a generalized quantity, and maintain continuity of meaning across different representations.

The results show that algebraic thinking cannot be reduced to symbolic procedural skills alone; rather, it constitutes a multidimensional mode of reasoning that requires the integrated development of structural sense, representation coordination, and generalization processes. These findings support contemporary theoretical perspectives emphasizing that algebraic thinking is a relational form of reasoning that emerges across different domains of mathematics.

One significant conclusion of the study is that geometry and measurement tasks provide a powerful context for making students' algebraic thinking processes visible. Students' tendency to use symbolic representations while explaining geometric relationships suggests that algebraic thinking develops not merely through abstract symbolic manipulation, but through the modeling of relationships among quantities. This finding indicates that treating algebra and geometry as independent domains may limit students' opportunities to construct relational meaning.

The study also revealed that students frequently relied on recursive strategies in their generalization processes and experienced difficulty when functional relationships were required. This suggests that limiting instruction to activities focused on continuing patterns may be insufficient for fostering variable-based thinking. Tasks grounded in the structural analysis of geometric representations may better support students' transition to generalized algebraic thinking.

From a pedagogical perspective, several implications can be drawn. Instructional processes may benefit from:

- addressing algebraic expressions not merely as computational tools, but as structures that explain mathematical relationships,
- designing tasks that promote meaningful transitions between different types of representations,
- incorporating problem situations in geometry and measurement contexts that make the idea of a variable explicit, and
- supporting generalization through activities that require far-term reasoning and the construction of functional relationships.

The theoretical contribution of this study lies in providing empirical evidence for cross-domain discussions of algebraic thinking by demonstrating, through qualitative analysis, how algebraic thinking emerges within geometry and measurement contexts. Methodologically, the study contributes to the literature by examining differences in algebraic thinking levels not only in terms of achievement, but also in terms of underlying thinking processes.

Nevertheless, the study was conducted with a specific grade level and a limited number of participants. Future research may investigate the developmental continuity of algebraic thinking across different grade levels, mathematical content areas, and longitudinal designs. In addition, intervention-based studies could provide more detailed evidence regarding how geometry-based tasks contribute to the development of algebraic thinking.

In conclusion, this study demonstrates that algebraic thinking plays a central role in students' processes of making sense of and generalizing mathematical relationships, and that geometry and measurement contexts offer valuable learning environments that support the development of this mode of reasoning.

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Data availability: The datasets generated and/or analyzed during the current study are not publicly available due to ethical and privacy considerations but are available from the corresponding author upon reasonable request.

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References

Apsari, R. A., Putri, R. I. I., Abels, M., & Prayitno, S. (2020). Geometry representation to develop algebraic thinking: A recommendation for a pattern investigation in pre-algebra class. *Journal on Mathematics Education*, *11*(1), 45-58. https://doi.org/10.22342/jme.11.1.9535.45-58

Blanton, M. L., & Kaput, J. J. (2011). Functional thinking as a route into algebra in the elementary grades. In J. Cai & E. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives* (pp. 5–23). Springer. https://doi.org/10.1007/978-3-642-17735-4_2

Blanton, M., Stephens, A., Knuth, E., Gardiner, A. M., Isler, I., & Kim, J. S. (2015). The development of children's algebraic thinking: The impact of a comprehensive early algebra intervention in third grade. *Journal for Research in Mathematics Education*, *46*(1), 39-87. https://doi.org/10.5951/jresematheduc.46.1.0039

Bozkuş, F., Kalaycı, Ö., & Toluk Uçar, Z. (2025). An analysis of algebra lesson: Can apples and pears be added? *Journal of Pedagogical Research, 9*(1), 274–295. https://doi.org/10.33902/JPR.202529564

Day, L., Siemon, D., Callingham, R., & Seah, R. (2024). Connecting the threads: The role of multiplicative thinking in algebraic, geometrical, and statistical reasoning. *Research in Mathematics Education, 26*(2), 325–347. https://doi.org/10.1080/14794802.2024.2372365

Elbir, D., & Özmen, Z. M. (2024). Investigation of mathematics teachers' teaching practices in transition to algebra in the context of letter symbols. *Journal of Pedagogical Sociology and Psychology, 6*(2), 47–63. https://doi.org/10.33902/jpsp.202426544

Ginsburg, H. P. (1997). *Entering the child's mind: The clinical interview in psychological research and practice*. Cambridge University Press.

Goldin, G. A. (2000). A scientific perspective on structured, task-based interviews in mathematics education research. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 517–545). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410602725

Grandau, L., & Stephens, A. C. (2006). Algebraic thinking and geometry. *Mathematics Teaching in the Middle School, 11*(7), 344–349. https://doi.org/10.5951/MTMS.11.7.0344

Jupri, A., Usdiyana, D., & Sispiyati, R. (2026). From student difficulties to design of a hypothetical learning trajectory: The case of system of linear equations in two variables. *International Journal of Didactical Studies, 7*(3), e41091. https://doi.org/10.33902/ijods.202641091

Kama, Z., Işıksal Bostan, M., & Tunç Pekkan, Z. (2023). Sixth-grade students' pattern generalization approaches. *Journal of Pedagogical Research, 7*(5), 136–155. https://doi.org/10.33902/JPR.202316928

Kaput, J. J. (2000). *Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum* (ED441664). ERIC. https://files.eric.ed.gov/fulltext/ED441664.pdf

Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema & T. A. Romberg (Eds.), *Mathematics classrooms that promote understanding* (pp. 133–155). Lawrence Erlbaum Associates. https://doi.org/10.4324/9781410602619

Kaput, J. J. (2008). What is algebra? What is algebraic reasoning? In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 235–272). Lawrence Erlbaum Associates.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*(3), 317–326. https://doi.org/10.1007/BF00311062

Kieran, C. (2004). Algebraic thinking in the early grades. In J. J. Kaput, D. W. Carraher, & M. L. Blanton (Eds.), *Algebra in the early grades* (pp. 139–151). Lawrence Erlbaum Associates.

Kieran, C. (2022). The multi-dimensionality of early algebraic thinking: Background, overarching dimensions, and new directions. *ZDM–Mathematics Education, 54*(6), 1131–1150. https://doi.org/10.1007/s11858-022-01435-6

Kuchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School, 7*(4), 23–26. https://www.jstor.org/stable/30213397

Kükey, E., & Aslaner, R. (2023). To what extent can teachers and preservice teachers predict students' mathematical thinking? A qualitative analysis. *Journal of Pedagogical Research, 7*(4), 91–110. https://doi.org/10.33902/JPR.202319993

Merriam, S. B., & Tisdell, E. J. (2016). *Qualitative research: A guide to design and implementation* (4th ed.). Jossey-Bass.

Miles, M. B., & Huberman, A. M. (1994). *Qualitative data analysis: An expanded sourcebook* (2nd ed.). Sage.

National Council of Teachers of Mathematics [NCTM]. (2000). *Principles and standards for school mathematics*. Author.

Nitko, A. J., & Brookhart, S. M. (2014). *Educational assessment of students* (6th ed.). Pearson.

Ntsohi, M. M. (2013). *Investigating teaching and learning of Grade 9 Algebra through excel spreadsheets: a mixed methods case study for Lesotho* [Doctoral dissertation, Stellenbosch University]. http://hdl.handle.net/10019.1/85657

Panasuk, R. M. (2010). Three phase ranking framework for assessing conceptual understanding in algebra using multiple representations. *Education, 131*(2), 235–257.

Popham, W. J. (2017). *Classroom assessment: What teachers need to know* (8th ed.). Pearson.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to students' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37-70. https://doi.org/10.1207/S15327833MTL0501_02

Rivera, F. D. (2010). Visual templates in pattern generalization activity. *Educational Studies in Mathematics, 73*(3), 297–328. https://doi.org/10.1007/s10649-009-9222-0

Sarmasági, P., Rumbus, A., Bilbao, J., Margitay-Becht, A., Pluhár, Z., Rebollar, C., & Dagienė, V. (2025). Bridging algebraic and computational thinking: Impacts on student development in K-12 education. *Informatics in Education, 24*(2), 343-376. https://doi.org/10.15388/infedu.2025.13

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification. *Educational Studies in Mathematics, 26*, 191–228. https://doi.org/10.1007/BF01273628

Sun, S., Sun, D., & Xu, T. (2023). The developmental progression of early algebraic thinking of elementary school students. *Journal of Intelligence, 11*(12), 222. https://doi.org/10.3390/jintelligence11120222

Tall, D., & Thomas, M. (1991). Encouraging versatile thinking in algebra using the computer. *Educational Studies in Mathematics, 22*, 125–147. https://doi.org/10.1007/BF00368352

Tanışlı, D., & Yavuzsoy Köse, N. (2011). Generalization strategies about linear figural patterns: Effect of figural and numerical clues. *Education and Science, 36*(160), 184–198. https://doi.org/10.15390/ES.2011.937

Türkmen, S., & Tanışlı, D. (2019). Seventh-grade students' views on the relationship between verbal and algebraic representations. *Elementary Education Online, 18*(3), 1298–1314. https://doi.org/10.14689/issn.2148-2624.1.7c1s.16m

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry*. University of Chicago.

Van de Walle, J., Karp, K., & Bay-Williams, J. (2012). *Elementary and middle school mathematics: Teaching developmentally* (8th ed.). Pearson.

Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Academic Press.